Logical foundations of categorization theory
Lecture 4

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Abstract

In this lecture, we show how formal contexts can be *enriched* with additional relations so as to make a (sound and complete) Kripke-style semantics of a basic normal *modal* logic of categories and concepts.

1 Basic normal modal logic of categories

Following the general methodology discussed in the previous lecture for interpreting the basic logic of categories on polarities, we are going to introduce a basic normal lattice-based modal logic, and interpret it on polarity-based (i.e. formal-context-based) relational structures.

Let Prop be a (countable or finite) set of atomic propositions. The language $\mathscr{L}$ of the *basic normal modal logic of formal concepts* is defined as follows:

$$\varphi := \bot \mid \top \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi$$

The *basic*, or *minimal normal modal $\mathscr{L}$-logic* is a set $\mathbf{L}$ of sequents $\varphi \vdash \psi$ (which intuitively read "φ is a subconcept of ψ") with $\varphi, \psi \in \mathscr{L}$, containing the following axioms:

- Sequents for propositional connectives:

$$p \vdash p, \qquad \bot \vdash p, \qquad p \vdash \top,$$
$$p \vdash p \vee q, \qquad q \vdash p \vee q, \qquad p \wedge q \vdash p, \qquad p \wedge q \vdash q,$$

- Sequents for modal operators:

$$\top \vdash \Box\top \qquad \Box p \wedge \Box q \vdash \Box(p \wedge q)$$
$$\Diamond\bot \vdash \bot \qquad \Diamond p \vee \Diamond q \vdash \Diamond(p \vee q)$$

and closed under the following inference rules:

$$\frac{\varphi \vdash \chi \quad \chi \vdash \psi}{\varphi \vdash \psi} \qquad \frac{\varphi \vdash \psi}{\varphi(\chi/p) \vdash \psi(\chi/p)} \qquad \frac{\chi \vdash \varphi \quad \chi \vdash \psi}{\chi \vdash \varphi \wedge \psi} \qquad \frac{\varphi \vdash \chi \quad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi}$$

$$\frac{\varphi \vdash \psi}{\Box\varphi \vdash \Box\psi} \qquad \frac{\varphi \vdash \psi}{\Diamond\varphi \vdash \Diamond\psi}$$

Intuitively, the modal fragment of **L** can be use e.g. to incorporate the viewpoints of individual agents into the syllogistic reasoning supported by the propositional fragment of **L**. By an *$\mathscr{L}$-logic* we understand any extension of **L** with $\mathscr{L}$-axioms $\varphi \vdash \psi$.

2 Polarity-based frames and their complex algebras

In order to endow the basic normal modal logic **L** defined above with relational semantics, in this section, we introduce relational structures $\mathbb{F} = (\mathbb{P}, R_\Box, R_\Diamond)$, such that $\mathbb{P} = (A, X, I)$ is a formal context, and $R_\Box$ and $R_\Diamond$ are suitable relations that respectively induce a normal box-type (i.e. completely meet-preserving) operation and a normal diamond-type (i.e. completely join-preserving) operation on the concept lattice $\mathbb{P}^+$. In order to guarantee that these operations are well defined on $\mathbb{P}^+$, we need to require some properties of $R_\Box$ and $R_\Diamond$, which we capture by the notion of *I-compatibility*, and for this, we will need some notation and facts which we collect in the next two sections.

2.1 Notation and basic properties

In what follows, we fix two sets A and X, and use a, b (resp. x, y) for elements of A (resp. X), and B, C, A_j (resp. Y, W, X_j) for subsets of A (resp. of X) throughout this section. For any relation $S \subseteq U \times W$, let

$$S^{(1)}[U'] := \{w \mid \forall u(u \in U' \Rightarrow uSw)\} \qquad S^{(0)}[W'] := \{u \mid \forall w(w \in W' \Rightarrow uSw)\}.$$

Well known properties of this construction (cf. [1, Sections 7.22-7.29]) are stated in the following lemma.

Lemma 1. *1. $B \subseteq C$ implies $S^{(1)}[C] \subseteq S^{(1)}[B]$, and $Y \subseteq W$ implies $S^{(0)}[W] \subseteq S^{(0)}[Y]$.*

2. $B \subseteq S^{(0)}[S^{(1)}[B]]$ and $Y \subseteq S^{(1)}[S^{(0)}[Y]]$.

3. $S^{(1)}[B] = S^{(1)}[S^{(0)}[S^{(1)}[B]]]$ and $S^{(0)}[Y] = S^{(0)}[S^{(1)}[S^{(0)}[Y]]]$.

4. $S^{(0)}[\bigcup \mathscr{Y}] = \bigcap_{Y \in \mathscr{Y}} S^{(0)}[Y]$ and $S^{(1)}[\bigcup \mathscr{B}] = \bigcap_{B \in \mathscr{B}} S^{(1)}[B]$.

Exercise 1. *Prove the statements of Lemma 1. Hint: for the first three items, use Exercise 4 and Example/Exercise 4 of Lecture 2.*

As we have done e.g. in Example/Exercise 4 of Lecture 2, for any formal context $\mathbb{P} = (A, X, I)$, we sometimes use $B^\uparrow$ for $I^{(1)}[B]$, and $Y^\downarrow$ for $I^{(0)}[Y]$, and say that B (resp. Y) is *Galois-stable* if $B = B^{\uparrow\downarrow}$ (resp. $Y = Y^{\downarrow\uparrow}$). When $B = \{a\}$ (resp. $Y = \{x\}$) we write $a^{\uparrow\downarrow}$ for $\{a\}^{\uparrow\downarrow}$ (resp. $x^{\downarrow\uparrow}$ for $\{x\}^{\downarrow\uparrow}$). The following lemma collects more well known facts (cf. [1, Sections 7.22-7.29]:

Lemma 2. *1. $B^{\uparrow}$ and $Y^{\downarrow}$ are Galois-stable.*

2. $B = \bigcup_{a\in B} a^{\uparrow\downarrow}$ and $Y = \bigcup_{y\in Y} y^{\downarrow\uparrow}$ for any Galois-stable B and Y.

3. Galois-stable sets are closed under arbitrary intersections.

Proof. Item (1) immediately follows from Lemma 1 (3). Let us show item 2. The inclusion $B \subseteq \bigcup_{a\in B} a^{\uparrow\downarrow}$ immediately follows from $a \in a^{\uparrow\downarrow}$ for any $a \in B$. Conversely, since B is Galois-stable, $a \in B$ implies $a^{\uparrow\downarrow} \subseteq B^{\uparrow\downarrow} = B$, hence $\bigcup_{a\in B} a^{\uparrow\downarrow} \subseteq B$, as required. The proof for Y is analogous. Item (3) immediately follows from Fact 1 and Exercise 4 of Lecture 2. ◻

2.2 I-compatible relations

Definition 1. *For any $\mathbb{P} = (A,X,I)$, any $S \subseteq A \times X$ (resp. $S \subseteq X \times A$) is I-*compatible *if $S^{(0)}[x]$ and $S^{(1)}[a]$ (resp. $S^{(0)}[a]$ and $S^{(1)}[x]$) are Galois-stable for all x and a.*

Item 3 of Lemma 1 immediately implies that I is an I-compatible relation.

Lemma 3. *For every polarity $\mathbb{P} = (A,X,I)$ and all $R \subseteq A \times X$ and $T \subseteq X \times A$,*

1. If R is I-compatible, then $R^{(0)}[Y] = R^{(0)}[Y^{\downarrow\uparrow}]$ and $R^{(1)}[B] = R^{(1)}[B^{\uparrow\downarrow}]$.

2. If T is I-compatible, then $T^{(1)}[Y] = T^{(1)}[Y^{\downarrow\uparrow}]$ and $T^{(0)}[B] = T^{(0)}[B^{\uparrow\downarrow}]$.

Proof. We only prove the first identity. By Lemma 1 (2), we have $Y \subseteq Y^{\downarrow\uparrow}$, which implies $R^{(0)}[Y^{\downarrow\uparrow}] \subseteq R^{(0)}[Y]$ by Lemma 1 (1). Conversely, if $a \in R^{(0)}[Y]$, i.e. $Y \subseteq R^{(1)}[a]$, then $Y^{\downarrow\uparrow} \subseteq (R^{(1)}[a])^{\downarrow\uparrow} = R^{(1)}[a]$, the last identity holding since R is I-compatible. Hence, $a \in R^{(0)}[Y^{\downarrow\uparrow}]$, as required. ◻

Exercise 2. *Prove the remaining identities of the lemma above.*

Lemma 4. *For every polarity $\mathbb{P} = (A,X,I)$ and all $R \subseteq A \times X$ and $T \subseteq X \times A$,*

1. If R is I-compatible and Y,B are Galois-stable, then $R^{(0)}[Y]$ and $R^{(1)}[B]$ are Galois-stable.

2. If T is I-compatible and Y,B are Galois-stable, then $T^{(0)}[B]$ and $T^{(1)}[Y]$ are Galois-stable.

Proof. We only prove that $R^{(0)}[Y]$ is Galois-stable. Since $Y = \bigcup_{y\in Y}\{y\}$, by Lemma 1 (4),

$$R^{(0)}[Y] = R^{(0)}[\bigcup_{y\in Y}\{y\}] = \bigcap_{y\in Y} R^{(0)}[\{y\}] = \bigcap_{y\in Y} R^{(0)}[y]. \tag{1}$$

By the I-compatibility of R, the last term is an intersection of Galois-stable sets, which is Galois-stable (cf. Lemma 2 (3)). ◻

Exercise 3. *Complete the proof of the lemma above.*

2.3 Enriched formal contexts

The following structures are generalizations of Kripke frames.

Definition 2. *An* enriched formal context *(or* polarity-based $\mathscr{L}$-frame*) is a tuple*

$$\mathbb{F} = (\mathbb{P}, R_\Box, R_\Diamond)$$

such that $\mathbb{P} = (A, X, I)$ *is a formal context, and* $R_\Box \subseteq A \times X$ *and* $R_\Diamond \subseteq X \times A$ *are* I-compatible *relations, that is,* $R_\Box^{(0)}[x]$ *(resp.* $R_\Diamond^{(0)}[a]$*) and* $R_\Box^{(1)}[a]$ *(resp.* $R_\Diamond^{(1)}[x]$*) are Galois-stable for all* $x \in X$ *and* $a \in A$*. The* complex algebra *of* $\mathbb{F}$ *is*

$$\mathbb{F}^+ = (\mathbb{P}^+, [R_\Box], \langle R_\Diamond \rangle),$$

where $\mathbb{P}^+$ *is the concept lattice of* $\mathbb{P}$*, and* $[R_\Box]$ *and* $\langle R_\Diamond \rangle$ *are unary operations on* $\mathbb{P}^+$ *defined as follows: for every* $c = ([\![c]\!], (\![c]\!)) \in \mathbb{P}^+$*,*

$$[R_\Box]c := (R_\Box^{(0)}[(\![c]\!)], (R_\Box^{(0)}[(\![c]\!)])^\uparrow) \quad \text{and} \quad \langle R_\Diamond \rangle c := ((R_\Diamond^{(0)}[[\![c]\!]])^\downarrow, R_\Diamond^{(0)}[[\![c]\!]]).$$

Lemma 4 and the I-compatibility of $R_\Box$ and $R_\Diamond$ ensure that the assignments above define operations $[R_\Box]$ and $\langle R_\Diamond \rangle$ on $\mathbb{P}^+$.

Lemma 5. *For any polarity-based frame* $\mathbb{F} = (\mathbb{P}, R_\Box, R_\Diamond)$*, the algebra* $\mathbb{F}^+ = (\mathbb{P}^+, [R_\Box], \langle R_\Diamond \rangle)$ *is a complete normal lattice expansion such that* $[R_\Box]$ *is completely meet-preserving and* $\langle R_\Diamond \rangle$ *is completely join-preserving.*

Proof. Let $R_\blacksquare \subseteq A \times X$ and $R_\blacklozenge \subseteq X \times A$ be defined as $aR_\blacksquare x$ iff $xR_\Diamond a$ and $xR_\blacklozenge a$ iff $aR_\Box x$. The I-compatibility of $R_\Box$ and $R_\Diamond$ immediately implies that $R_\blacksquare$ and $R_\blacklozenge$ are also I-compatible. Hence, by Lemma 4, the following assignments define operations $[R_\blacksquare]$ and $\langle R_\blacklozenge \rangle$ on $\mathbb{P}^+$: for every $c = ([\![c]\!], (\![c]\!)) \in \mathbb{P}^+$,

$$[R_\blacksquare]c := (R_\blacksquare^{(0)}[(\![c]\!)], (R_\blacksquare^{(0)}[(\![c]\!)])^\uparrow) \quad \text{and} \quad \langle R_\blacklozenge \rangle c := ((R_\blacklozenge^{(0)}[[\![c]\!]])^\downarrow, R_\blacklozenge^{(0)}[[\![c]\!]]).$$

Since $\mathbb{P}^+$ is a complete lattice, by [1, Proposition 7.31], to show that $[R_\Box]$ is completely meet-preserving and $\langle R_\Diamond \rangle$ is completely join-preserving, it is enough to show that $\langle R_\blacklozenge \rangle$ is the left adjoint of $[R_\Box]$ and that $[R_\blacksquare]$ is the right adjoint of $\langle R_\Diamond \rangle$. For any $c, d \in \mathbb{P}^+$,

$$\begin{array}{llll} \langle R_\blacklozenge \rangle c \leq d & \text{iff} & (\![d]\!) \subseteq R_\blacklozenge^{(0)}[[\![c]\!]] & \text{ordering of concepts} \\ & \text{iff} & (\![d]\!) \subseteq R_\Box^{(1)}[[\![c]\!]] & \text{(definition of } R_\blacklozenge) \\ & \text{iff} & [\![c]\!] \subseteq R_\Box^{(0)}[(\![d]\!)] & \text{(Galois connection)} \\ & \text{iff} & c \leq [R_\Box]d. & \text{ordering of concepts} \end{array}$$

Likewise, one shows that $[R_\blacksquare]$ is the right adjoint of $\langle R_\Diamond \rangle$. □

Exercise 4. *Prove that* $[R_\blacksquare]$ *is the right adjoint of* $\langle R_\Diamond \rangle$*.*

Interpretation of modal formulas in polarity-based frames. For any enriched formal context $\mathbb{F} = (\mathbb{P}, R_\Box, R_\Diamond)$, a *valuation* on $\mathbb{F}$ is a map $V : \mathsf{Prop} \to \mathbb{F}^+$. A *polarity-based $\mathscr{L}$-model* is a tuple $\mathbb{M} = (\mathbb{F}, V)$. For every enriched formal context $\mathbb{F} = (\mathbb{P}, R_\Box, R_\Diamond)$, any valuation V on $\mathbb{F}$ extends to an interpretation map of $\mathscr{L}$-formulas defined as follows:

$$
\begin{array}{rcl}
V(p) & = & ([\![p]\!], (\![p]\!)) \\
V(\top) & = & (A, A^{\uparrow}) \\
V(\bot) & = & (X^{\downarrow}, X) \\
V(\varphi \wedge \psi) & = & ([\![\varphi]\!] \cap [\![\psi]\!], ([\![\varphi]\!] \cap [\![\psi]\!])^{\uparrow}) \\
V(\varphi \vee \psi) & = & (((\![\varphi]\!) \cap (\![\psi]\!))^{\downarrow}, (\![\varphi]\!) \cap (\![\psi]\!)) \\
V(\Box\varphi) & = & (R_\Box^{(0)}[(\![\varphi]\!)], (R_\Box^{(0)}[(\![\varphi]\!)])^{\uparrow}) \\
V(\Diamond\varphi) & = & ((R_\Diamond^{(0)}[[\![\varphi]\!]])^{\downarrow}, R_\Diamond^{(0)}[[\![\varphi]\!]])
\end{array}
$$

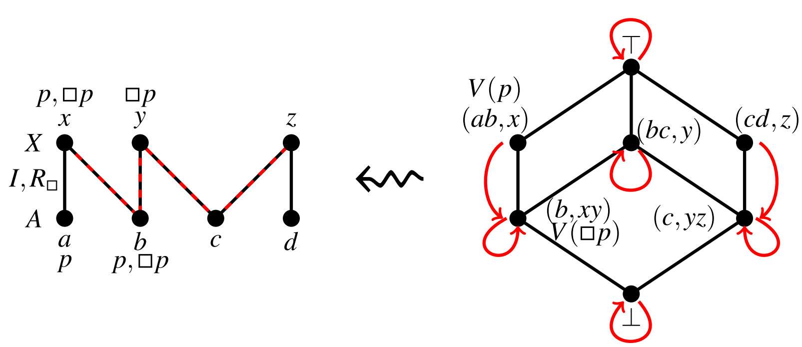
As we discussed in the previous lecture, the homomorphic extension of each valuation gives rise to the recursive definition of the relations of "membership" $\Vdash$ of objects in categories, and of features "describing" categories ($\succ$) extended to all $\mathscr{L}$-formulas, and hence also to modal formulas. Hence, spelling out the definition of the homomorphic extension of a given assignment on the complex algebra of a polarity-based frame according to the following conditions:

$$
\mathbb{M}, a \Vdash \varphi \quad \text{iff} \quad a \in [\![\varphi]\!]_{\mathbb{M}} \qquad \mathbb{M}, x \succ \varphi \quad \text{iff} \quad x \in (\![\varphi]\!)_{\mathbb{M}}
$$

yields the following recursive definition of the "membership relation" $\Vdash$ of objects in categories, and of features "describing" categories ($\succ$) extended to the interpretation of modal $\mathscr{L}$-formulas:

$$
\begin{array}{lll}
\mathbb{M}, a \Vdash \Box\varphi & \text{iff} & \text{for all } x \in X, \text{ if } \mathbb{M}, x \succ \varphi, \text{ then } aR_\Box x \\
\mathbb{M}, x \succ \Box\varphi & \text{iff} & \text{for all } a \in A, \text{ if } \mathbb{M}, a \Vdash \Box\varphi, \text{ then } aIx. \\
\mathbb{M}, a \Vdash \Diamond\varphi & \text{iff} & \text{for all } x \in X, \text{ if } \mathbb{M}, x \succ \Diamond\varphi, \text{ then } aIx \\
\mathbb{M}, x \succ \Diamond\varphi & \text{iff} & \text{for all } a \in A, \text{ if } \mathbb{M}, a \Vdash \varphi, \text{ then } xR_\Diamond a.
\end{array}
$$

Thus, in each model, $\Box\varphi$ is interpreted as the concept whose members are those objects which are $R_\Box$-related to every feature in the description of φ, and $\Diamond\varphi$ is interpreted as the category described by those features which are $R_\Diamond$-related to every member of φ. To illustrate this with a concrete example, consider the enriched formal context represented on the left hand side of the picture below (for simplicity's sake $R_\Diamond$ is not represented, and the black and red dashed lines refer to elements that are both I-related and $R_\Box$-related):



Finally, as to the interpretation of sequents:

$$\begin{array}{llllll}\mathbb{M} \models \varphi \vdash \psi & \text{iff} & \llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket & \text{iff} & \text{for all } a \in A, \text{ if } \mathbb{M}, a \Vdash \varphi, \text{ then } \mathbb{M}, a \Vdash \psi \\ & \text{iff} & (\![\psi]\!) \subseteq (\![\varphi]\!) & \text{iff} & \text{for all } x \in X, \text{ if } \mathbb{M}, x \succ \psi, \text{ then } \mathbb{M}, x \succ \varphi.\end{array}$$

A sequent $\varphi \vdash \psi$ is *valid* on an enriched formal context $\mathbb{F}$ (in symbols: $\mathbb{F} \models \varphi \vdash \psi$) if $\mathbb{M} \models \varphi \vdash \psi$ for every model $\mathbb{M}$ based on $\mathbb{F}$.

3 Soundness and completeness

In the present section we prove the following

Proposition 1. *The basic normal modal logic of formal concepts is sound and complete w.r.t. the class of polarity-based frames.*

3.1 Soundness

Proposition 2. *For any polarity-based model $\mathbb{M}$,*

1. *if $\mathbb{M} \models \varphi \vdash \psi$, then $\mathbb{M} \models \Box\varphi \vdash \Box\psi$ and $\mathbb{M} \models \Diamond\varphi \vdash \Diamond\psi$;*
2. *$\mathbb{M} \models \top \vdash \Box\top$ and $\mathbb{M} \models \Diamond\bot \vdash \bot$;*
3. *$\mathbb{M} \models \Box\varphi \wedge \Box\psi \vdash \Box(\varphi \wedge \psi)$ and $\mathbb{M} \models \Diamond(\varphi \vee \psi) \vdash \Diamond\varphi \vee \Diamond\psi$.*

Proof. We only prove the statements relative to $\Box$-formulas. If $\mathbb{M} \models \varphi \vdash \psi$, then $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket$ and $(\![\psi]\!) \subseteq (\![\varphi]\!)$, which implies, by Lemma 1 (1), that

$$\llbracket \Box\varphi \rrbracket = R_{\Box}^{(0)}[(\![\varphi]\!)] \subseteq R_{\Box}^{(0)}[(\![\psi]\!)] = \llbracket \Box\psi \rrbracket,$$

which proves item (1). As to item (2), it is enough to show that $A = \llbracket \top \rrbracket \subseteq \llbracket \Box\top \rrbracket = R_{\Box}^{(0)}[(\![\top]\!)] = R_{\Box}^{(0)}[A^{\uparrow}]$. By adjunction, it is enough to show that $A^{\uparrow} \subseteq R_{\Box}^{(1)}[A]$. Since $R_{\Box}$ is I-compatible and A is Galois-stable, Lemma 4 (1) implies that $R_{\Box}^{(1)}[A]$ is Galois-stable, hence it is enough to show that $A^{\uparrow} \subseteq (R_{\Box}^{(1)}[A])^{\downarrow\uparrow}$. For this, it is enough to show that $(R_{\Box}^{(1)}[A])^{\downarrow} \subseteq A$, which is certainly the case. As to item (3),

$$\begin{array}{lcll}\llbracket \Box(\varphi) \wedge \Box(\psi) \rrbracket & = & \llbracket \Box(\varphi) \rrbracket \cap \llbracket \Box(\psi) \rrbracket & \text{definition of } \llbracket\cdot\rrbracket \text{ on } \wedge\text{-formulas} \\ & = & R^{(0)}[(\![\varphi]\!)] \cap R_{\Box}^{(0)}[(\![\psi]\!)] & \text{definition of } \llbracket\cdot\rrbracket \text{ on } \Box\text{-formulas} \\ & = & R_{\Box}^{(0)}[(\![\varphi]\!) \cup (\![\psi]\!)] & \text{Lemma 1 (4)} \\ & = & R_{\Box}^{(0)}[((\![\varphi]\!) \cup (\![\psi]\!))^{\downarrow\uparrow}] & \text{Lemma 3} \\ & = & R_{\Box}^{(0)}[((\![\varphi]\!)^{\downarrow} \cap (\![\psi]\!)^{\downarrow})^{\uparrow}] & \text{Lemma 1 (4)} \\ & = & R_{\Box}^{(0)}[(\llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket)^{\uparrow}] & V(\varphi), V(\psi) \text{ formal concepts} \\ & = & R_{\Box}^{(0)}[(\llbracket \varphi \wedge \psi \rrbracket)^{\uparrow}] & \text{definition of } \llbracket\cdot\rrbracket \text{ on } \wedge\text{-formulas} \\ & = & R_{\Box}^{(0)}[(\![\varphi \wedge \psi]\!)] & \text{definition of } (\![\cdot]\!) \\ & = & \llbracket \Box(\varphi \wedge \psi) \rrbracket. & \text{definition of } \llbracket\cdot\rrbracket \text{ on } \Box\text{-formulas}\end{array}$$

$\Box$

Exercise 5. *Complete the proof of the proposition above.*

3.2 Completeness

The completeness of **L** can be proven via a standard canonical model construction. For any lattice with normal operators $(\mathbb{L}, \Box, \Diamond)$, let $\mathbb{F}_{\mathbb{L}} = (\mathbb{P}_{\mathbb{L}}, R_\Box, R_\Diamond)$ be defined as follows: $\mathbb{P}_{\mathbb{L}} = (A, X, I)$ where A (resp. X) is the set of lattice filters (resp. ideals) of $\mathbb{L}$, and aIx iff $a \cap x \neq \varnothing$. Moreover, let $R_\Box \subseteq A \times X$ and $R_\Diamond \subseteq X \times A$ be defined as follows:

$$\begin{array}{lll} aR_\Box x & \text{iff} & \Box u \in a \text{ for some } u \in \mathbb{L} \text{ such that } u \in x \\ xR_\Diamond a & \text{iff} & \Diamond u \in x \text{ for some } u \in \mathbb{L} \text{ such that } u \in a. \end{array}$$

In what follows, for any $a \in A$ and $x \in X$, we let

$$\begin{array}{ll} \Box x := \{\Box u \in \mathbb{L} \mid u \in x\} & \Box^{-1} a := \{u \in \mathbb{L} \mid \Box u \in a\} \\ \Diamond a := \{\Diamond u \in \mathbb{L} \mid u \in a\} & \Diamond^{-1} x := \{u \in \mathbb{L} \mid \Diamond u \in x\}. \end{array}$$

Lemma 6. *For $\mathbb{F}_{\mathbb{L}} = (\mathbb{P}_{\mathbb{L}}, R_\Box, R_\Diamond)$ as above, and any $a \in A$ and $x \in X$,*

1. $R_\Box^{(0)}[x] = \{b \in A \mid b \cap \Box x \neq \varnothing\}$ *and* $R_\Diamond^{(0)}[a] = \{y \in X \mid y \cap \Diamond a \neq \varnothing\}$;
2. $R_\Box^{(1)}[a] = \{y \in X \mid y \cap \Box^{-1} a \neq \varnothing\}$ *and* $R_\Diamond^{(1)}[x] = \{b \in A \mid b \cap \Diamond^{-1} x \neq \varnothing\}$.
3. $\top \in \Box^{-1} a \neq \varnothing$ *and* $\bot \in \Diamond^{-1} x \neq \varnothing$.

Exercise 6. *Prove Lemma 6. Hint for item (3): use that $\Box\top = \top$ and $\Diamond\bot = \bot$.*

Lemma 7. *For $\mathbb{F}_{\mathbb{L}}$ as above, and any $a \in A$ and $x \in X$,*

1. $(R_\Box^{(0)}[x])^\uparrow = \{y \in X \mid \Box x \subseteq y\}$ *and* $(R_\Diamond^{(0)}[a])^\downarrow = \{b \in A \mid \Diamond a \subseteq b\}$;
2. $(R_\Box^{(1)}[a])^\downarrow = \{b \in A \mid \Box^{-1} a \subseteq b\}$ *and* $(R_\Diamond^{(1)}[x])^\downarrow = \{y \in X \mid \Diamond^{-1} x \subseteq y\}$;
3. $(R_\Box^{(0)}[x])^{\uparrow\downarrow} = \{b \in A \mid b \cap \Box x \neq \varnothing\} = R_\Box^{(0)}[x]$ *and* $(R_\Diamond^{(0)}[a])^{\downarrow\uparrow} = \{y \in X \mid y \cap \Diamond a \neq \varnothing\} = R_\Diamond^{(0)}[a]$;
4. $(R_\Box^{(1)}[a])^{\downarrow\uparrow} = \{y \in X \mid y \cap \Box^{-1} a \neq \varnothing\} = R_\Box^{(1)}[a]$ *and* $(R_\Diamond^{(1)}[x])^{\uparrow\downarrow} = \{b \in A \mid b \cap \Diamond^{-1} x \neq \varnothing\} = R_\Diamond^{(1)}[x]$.

Proof. We only sketch the proof of the identities about $R_\Box$. Items (1) and (2) readily follow from Lemma 6 (1) and (2). As to items (3) and (4), from the previous items it immediately follows that $(R_\Box^{(0)}[x])^{\uparrow\downarrow} = \{b \in A \mid \lceil \Box x \rceil \cap b \neq \varnothing\}$ and $(R_\Box^{(1)}[a])^{\downarrow\uparrow} = \{y \in X \mid \lfloor \Box^{-1} a \rfloor \cap y \neq \varnothing\}$, where $\lceil \Box x \rceil$ and $\lfloor \Box^{-1} a \rfloor$ respectively denote the ideal generated by $\Box x$ and the filter generated by $\Box^{-1} a$. Then, using the monotonicity of $\Box$, and that any $x \in X$ is closed under finite joins and any $b \in A$ is upward-closed, one can show that $\{b \in A \mid \lceil \Box x \rceil \cap b \neq \varnothing\} = \{b \in A \mid \Box x \cap b \neq \varnothing\} = R_\Box^{(0)}[x]$, and using the meet-preservation of $\Box$, one can show that $\{y \in X \mid \lfloor \Box^{-1} a \rfloor \cap y \neq \varnothing\} = \{y \in X \mid \Box^{-1} a \cap y \neq \varnothing\} = R_\Box^{(1)}[a]$, as required. Notice that the last equality holds for every $a \in A$ under the assumption that $\Box^{-1} a \neq \varnothing$, which, by Lemma 6 (3), is guaranteed by $\Box$ being normal. ∎

Exercise 7. *Complete the proof of the lemma above.*

Items (3) and (4) of the lemma above immediately imply that:

Corollary 1. *$\mathbb{F}_{\mathbb{L}}$ is an enriched formal context (cf. Definition 2).*

Lemma 8. *For $\mathbb{F}_{\mathbb{L}}$ as above, and any $a \in A$ and $x \in X$,*

1. *If x is the ideal generated by some $u \in \mathbb{L}$, then $R_{\Box}^{(0)}[x] = \{a \in A \mid \Box u \in a\}$.*
2. *If a is the filter generated by some $u \in \mathbb{L}$, then $R_{\Diamond}^{(0)}[a] = \{x \in X \mid \Diamond u \in x\}$.*

Proof. By Lemma 6, $aR_{\Box}x$ iff $a \in R_{\Box}^{(0)}[x]$ iff $a \cap \Box x \neq \varnothing$. By assumption, x is the ideal generated by u, hence u is the greatest element of x; so the monotonicity of $\Box$ implies that $\Box u$ is the greatest element of $\Box x$. Since a is a filter, and hence is upward-closed, $a \cap \Box x \neq \varnothing$ is equivalent to $\Box u \in a$, which completes the proof. □

Exercise 8. *Complete the proof of the lemma above.*

The *canonical enriched formal context* is defined by instantiating the construction above to the Lindembaum-Tarski algebra of **L**. In this case, let V be the valuation such that $[\![p]\!]$ (resp. $(\![p]\!)$) is the set of the filters (resp. ideals) to which p belongs, and let $\mathbb{M} = (\mathbb{F}_{\mathbf{L}}, V)$ be the canonical model. Then the following holds for $\mathbb{M}$:

Lemma 9 (Truth lemma)**.** *For every $\varphi \in \mathscr{L}$,*

$$\mathbb{M}, a \Vdash \varphi \text{ iff } \varphi \in a \qquad \text{and} \qquad \mathbb{M}, x \succ \varphi \text{ iff } \varphi \in x.$$

Proof. By induction on φ. We only show the inductive step for $\varphi := \Box\sigma$.

	$\mathbb{M}, a \Vdash \Box\sigma$	
iff	$a \in R_{\Box}^{(0)}[(\![\sigma]\!)]$	definition of $[\![\Box\sigma]\!]$
iff	$a \in R_{\Box}^{(0)}[\{x \mid \sigma \in x\}]$	induction hypothesis
iff	for all $x \in X$, if $\sigma \in x$ then $a \cap \Box x \neq \varnothing$	definition of $R_{\Box}$
iff	$a \cap \Box\lceil\sigma\rceil \neq \varnothing$	$\lceil\sigma\rceil$ is the smallest $x \in X$ s.t. $\sigma \in x$
iff	$\Box\sigma \in a$.	a upward-closed, and $\Box\sigma$ is the greatest el. in $\Box\lceil\sigma\rceil$

	$\mathbb{M}, x \succ \Box\sigma$	
iff	$x \in (\![\Box\sigma]\!)$	
iff	$x \in [\![\Box\sigma]\!]^{\uparrow}$	
iff	$x \in (\{a \in A \mid \Box\sigma \in a\})^{\uparrow}$	proof above
iff	for all $a \in A$, if $\Box\sigma \in a$ then $x \cap a \neq \varnothing$	
iff	$x \cap \lfloor\Box\sigma\rfloor \neq \varnothing$	$\lfloor\Box\sigma\rfloor$ is the smallest $a \in A$ s.t. $\Box\sigma \in a$
iff	$\Box\sigma \in x$.	x downward-closed, and $\Box\sigma$ is the smallest el. in $\lfloor\Box\sigma\rfloor$

□

Exercise 9. *Complete the proof of the truth lemma.*

Proposition 3 (Completeness)**.** *If $\varphi \vdash \psi$ is an $\mathscr{L}$-sequent which is not derivable in* **L***, then $\mathbb{M} \not\models \varphi \vdash \psi$.*

Proof. If the $\mathscr{L}$-sequent $\varphi \vdash \psi$ is not derivable in $\mathbf{L}$, then $a \cap x = \varnothing$, where a denotes the filter in the Lindenbaum-Tarski algebra generated by φ and x denotes the ideal in the Lindenbaum-Tarski algebra generated by ψ. Then the Truth lemma implies that $a \in [\![\varphi]\!]$ and $a \notin [\![\psi]\!]$, hence $[\![\varphi]\!] \not\subseteq [\![\psi]\!]$, i.e. $\mathbb{M} \not\models \varphi \vdash \psi$, as required. □

References

[1] B. Davey and H. Priestley. *Introduction to lattices and order*. Cambridge univ. press, 2002.