Logical foundations of categorization theory
Lecture 5

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Abstract

In this lecture, based on [7, Section 4], we discuss possible interpretations of the basic normal modal logic of categories, and related to this, some of its axiomatic extensions, and time-permitting, the first-order correspondents on polarity-based frames of these axioms and their relation with their classical counterparts, both conceptual and technical.

1 Recap: from FCA to the logic of categories

As discussed in Lecture 3, the interpretive strategy supported by the polarity-based semantics drops the interpretation of $\wedge$ and $\vee$ as conjunction and disjunction in natural language and stipulates that formulas do not denote sentences describing states of affairs, but rather, denote objects with a different ontology, such as categories, concepts, questions, theories, to which a truth value might not necessarily be applicable.

Polarities as abstract databases. The idea that lattices are the proper mathematical environment for discussing "especially systems which are in any sense hierarchies" goes back to Birkhoff [3]. Based on this idea, Wille [8] and his collaborators developed Formal Concept Analysis (FCA) as a theory in information science aimed at the formal representation and analysis of conceptual structures, which has been applied to a wide range of fields ranging from psychology, sociology, and linguistics to biology and chemistry.

Building on philosophical insights developed by the school of Port-Royal [1], Wille specified concepts in terms of their *extension*, i.e. the set of objects which exemplify the given concept, and their *intension*, i.e. the set of attributes shared by the objects in the extension of the given concept, and identified Birkhoff's polarities $\mathbb{P} = (A, X, I)$ (aka *formal contexts* cf. Lecture 1), as the appropriate mathematical environment in which these ideas could be formally represented: indeed, as we discussed e.g. in Lecture 3, a polarity $\mathbb{P}$ as above can be understood as an abstract representation of a *database*, recording information about a given set A of *objects* (relevant to a given context or situation), and a set X of relevant attributes or *features*. In this representation, the (incidence) relation $I \subseteq A \times X$ encodes whether object $a \in A$ has feature $x \in X$ as aIx. The

Galois-adjoint pair of maps $(\cdot)^{\uparrow} : \mathscr{P}(A) \to \mathscr{P}(X)$ and $(\cdot)^{\downarrow} : \mathscr{P}(X) \to \mathscr{P}(A)$ can be understood as *concept-generating maps*: namely, as maps taking any set B of objects to the intension $B^{\uparrow}$ which uniquely determines the formal concept $(B^{\uparrow\downarrow}, B^{\uparrow})$ generated by B, and any set Y of attributes to the extension $Y^{\downarrow}$ which uniquely determines the formal concept $(Y^{\downarrow}, Y^{\downarrow\uparrow})$ generated by Y. Hence, the philosophical and cognitive insight that concepts do not occur in isolation, but rather arise within a hierarchy of other concepts, finds a very natural representation in the construction of the complete lattice $\mathbb{P}^{+}$ and its natural order as the *sub-concept* relation. Indeed, a subconcept of a given concept, understood as a more restrictive concept, will have a smaller extension (i.e. fewer examples) and a larger intension (i.e. a larger set of requirements that objects need to satisfy in order to count as examples of the given sub-concept). We saw that this interpretation accounts for the failure of distributivity.

Propositional lattice logic as the basic logic of formal concepts. Imposing the FCA interpretation of polarities discussed above on the polarity-based semantics of the basic (normal modal) logic **L** discussed in Lectures 3 and 4 yields an interpretation of $\mathscr{L}$-formulas as terms (i.e. names) denoting formal concepts. Starting from assignments to proposition variables, any $\mathscr{L}$-formula φ is then interpreted on a given polarity $\mathbb{P} = (A, X, I)$ as a formal concept $([\![\varphi]\!], (\![\varphi]\!)) \in \mathbb{P}^{+}$; specifically, for each object $a \in A$ and feature $x \in X$, the relations $a \Vdash \varphi$ and $x \succ \varphi$ can be respectively understood as 'object a is a member of (or exemplifies) concept φ' and 'feature x *describes* concept φ', in the sense that x is a required attribute of every example/member of φ. Accordingly, this reading suggests that $\varphi \wedge \psi$ can be understood as 'the greatest (i.e. least restrictive) common subconcept of concept φ and concept ψ', i.e. the concept the extension of which is the intersection of the extensions of φ and ψ. Similarly, $\varphi \vee \psi$ is 'the least (i.e. most restrictive) common superconcept of concept φ and concept ψ', i.e. the concept the intension of which is the intersection of the intensions of φ and ψ; the constant $\top$ can be understood as the most generic (or comprehensive) concept (i.e. the one that, when interpreted in any given polarity $\mathbb{P}$ as above, allows all objects $a \in A$ as examples) while $\bot$ as the most restrictive (i.e. the one that, when interpreted in any given polarity $\mathbb{P}$ as above, requires its examples to have all attributes $x \in X$). Finally, $\varphi \vdash \psi$ can be understood as the statement that 'concept φ is a sub-concept of concept ψ'.

As mentioned above, this interpretation accounts for the failure of distributivity. Indeed, objects in the extension of concept $\varphi \vee \psi$ are only required to have all attributes common to the intentions of concepts φ and ψ; this weaker requirement potentially allows objects in $[\![\varphi \vee \psi]\!]$ which belong to neither $[\![\varphi]\!]$ nor to $[\![\psi]\!]$. To illustrate this point concretely, in lecture 3 we discussed the example of a polarity representing a 'database' of theatrical plays, and we motivated the failure of distributivity in the context of that example.

2 From semantics to meaning

Lattice-based normal modal logic as an epistemic logic of formal concepts. So far, we have discussed how the polarity-based semantics of the basic propositional lattice logic **L** allows for an interpretation of $\mathscr{L}$-formulas as names of formal concepts, and for a coherent interpretation of the meaning of all propositional lattice connectives so that the failure of distributivity becomes essential to capturing 'correct reasoning' in the context of conceptual hierarchies. Next, based on [5, 6], we discuss how this interpretation can be extended also to the modal connectives. For the sake of simplicity, let us consider the $\Diamond$-free fragment $\mathbf{L}_\Box$ of the basic normal modal logic $\mathbf{L}$ introduced in Lecture 4. In what follows, we will abuse notation and identify $\mathbf{L}_\Box$ with its language. As discussed in Lecture 4, this logic can be interpreted on relational structures $\mathbb{F} = (\mathbb{P}, R_\Box)$ such that $\mathbb{P} = (A, X, I)$ is a polarity, and $R_\Box \subseteq A \times X$ is an I-compatible relation such that, for any assignment $v : \mathsf{Prop} \to \mathbb{P}^+$, corresponding relations $\Vdash \subseteq A \times \mathbf{L}_\Box$ and $\succ \subseteq X \times \mathbf{L}_\Box$ can be defined. In the case of $\Box$-formulas, this yields

$$\begin{array}{lll} a \Vdash \Box\varphi & \text{iff} & \text{for all } x \in X, \text{ if } x \succ \varphi \text{ then } aR_\Box x \\ x \succ \Box\varphi & \text{iff} & \text{for all } a \in A, \text{ if } a \Vdash \Box\varphi \text{ then } aIx. \end{array}$$

Building on the understanding of polarities as abstract representation of databases, the relational structures $\mathbb{F} = (\mathbb{P}, R_\Box)$ can be understood as (abstract representations of) databases which not only encode objective information about objects and their features (by way of the incidence relation I of $\mathbb{P}$), but also encode *subjective* information regarding whether given objects have given attributes *according to a given agent*; this understanding allows us to read $aR_\Box x$ as 'object a has attribute x *according to agent i*. Of course, this interpretation can be further specialized so as to represent agents' knowledge ($aR_\Box x$ iff 'agent i knows that object a has attribute x'), beliefs ($aR_\Box x$ iff 'agent i believes that object a has attribute x'), perceptions ($aR_\Box x$ iff 'agent i sees that object a has attribute x'), evidential reasoning ($aR_\Box x$ iff 'agent i has evidence that object a has attribute x'), and so on. Each of these epistemic interpretations will give rise to a different epistemic reading of $\Box\varphi$ as 'concept φ according to the given agent i': namely, 'concept φ as is known/believed/perceived/experienced by agent i'. Also in the case of $\mathbf{L}_\Box$-formulas, for every object a and attribute x, the symbols $a \Vdash \Box\varphi$ and $x \succ \Box\varphi$ can be understood as 'object a is a member/example of $\Box\varphi$' and 'attribute x describes $\Box\varphi$', respectively. Interestingly, the condition that

$$a \Vdash \Box\varphi \quad \text{iff} \quad \text{for all } x \in X, \text{ if } x \succ \varphi \text{ then } aR_\Box x$$

can then be informally understood as saying that any object a is a member/example of concept φ according to agent i if and only if agent i attributes to a all the defining features of concept φ. This reading is indeed coherent with our informal understanding of which objects should count as members of 'concept φ according to agent i'.

Exercise 1. *Propose an informal understanding, along the lines just discussed, of the following interpretation clauses:*

1. $\mathbb{M}, x \succ \Box\varphi$ *iff for all* $a \in A$, *if* $\mathbb{M}, a \Vdash \Box\varphi$, *then* aIx.

2. $\mathbb{M}, a \Vdash \Diamond\varphi$ *iff for all* $x \in X$, *if* $\mathbb{M}, x \succ \Diamond\varphi$, *then* aIx

3. $\mathbb{M}, x \succ \Diamond\varphi$ *iff for all* $a \in A$, *if* $\mathbb{M}, a \Vdash \varphi$, *then* $xR_\Diamond a$.

Finally, one would also expect that the different variants of epistemic interpretations would satisfy different axioms; for instance, if $\Box\varphi$ is interpreted as 'concept φ

as is *known* by agent i', one would ask whether there is some $\mathbf{L}_\Box$-axioms which would encode the counterparts, in the lattice-based setting, of well known classical epistemic principles such as the *factivity* condition which distinguishes knowledge from belief, and what would this condition look like in the context of polarity-based relational structures. As is well known, in the setting of classical normal modal logic, factivity is formalized as the modal axiom $\Box\varphi \to \varphi$ (which reads 'if agent i knows that φ, then φ is indeed the case'). Moreover, this is a Sahlqvist formula and corresponds on Kripke frames (W,R) to R being reflexive or, equivalently, to $\Delta \subseteq R$. Since $\mathbf{L}_\Box$ is based on sequents and not on formulas, the closest approximation to the classical formula $\Box\varphi \to \varphi$ is the $\mathbf{L}_\Box$-sequent $\Box\varphi \vdash \varphi$, which turns out (cf. [4, Proposition 4.3]) to correspond on polarity-based structures $\mathbb{F}$ as above to the first-order condition $R_\Box \subseteq I$. That is, for every object a and feature x, if $aR_\Box x$ (i.e. if a is endowed with x according to agent i) then aIx (i.e. object a indeed has feature x).

Exercise 2. *Prove that for every polarity-based frame $\mathbb{F} = (\mathbb{P}, R_\Box)$,*

$$\mathbb{F} \models \Box p \vdash p \quad \textit{iff} \quad \forall a \forall x(aR_\Box x \Rightarrow aIx).$$

Hint: from right to left, fix a valuation V on $\mathbb{P}^+$, and show that, for every $a \in A$, if $a \in [\![\Box p]\!]_V$ then $a \in [\![p]\!]_V$; for the converse direction, assume that $aR_\Box x$ but not aIx for some $a \in A$ and $x \in X$, and find an assignment V on $\mathbb{P}^+$ such that $\mathbb{F} \not\models \Box p \vdash p$.

This condition is arguably an appropriate rendering of factivity in the setting of polarity-based relational structures, which suggests that more modal epistemic principles might retain their intended interpretation even under a substantial generalization step such as the one from the classical (i.e. Boolean) to the lattice-based setting. Indeed, this is also the case for *positive introspection*, which in the language of classical modal logic is formalized as $\Box\varphi \to \Box\Box\varphi$ (which reads 'if agent i knows that φ, then agent i knows that she knows φ'). As is well known, this axiom is a Sahlqvist formula the first-order correspondent of which on Kripke frames (W,R) is the condition $R \circ R \subseteq R$, concisely expressing that the relation R is transitive. Again, the $\mathbf{L}_\Box$-sequent $\Box\varphi \vdash \Box\Box\varphi$ turns out (cf. [4, Proposition 4.3]) to correspond on polarity-based structures $\mathbb{F}$ to the first-order condition[1] that reads: for every object a and feature x, if agent i thinks that a has feature x, then (agent i must recognize a as an example of what i understands *x-objects* to be, i.e. as a member of i's understanding of the formal concept generated by feature x, and hence) agent i must attribute to a also all the features that, according to i, are shared by all x-objects.

Exercise 3. *Prove that for every polarity-based frame $\mathbb{F} = (\mathbb{P}, R_\Box)$,*

$$\mathbb{F} \models \Box p \vdash \Box\Box p \quad \textit{iff} \quad R_\Box \subseteq R_\Box \,;_I R_\Box.$$

Additional details and hints are collected in the handout.

As in the case of factivity, one can argue that this condition is an appropriate rendering of the principle of positive introspection in the setting of polarity-based relational

[1] With the aid of the notation $;_I$ for relational composition modulo the polarity relation I (cf. [4, Section 3.4] for the full definition) this condition can be succinctly captured as $R_\Box \subseteq R_\Box \,;_I R_\Box$.

structures, since it is clearly an internal coherence requirement which seeks to justify any given attribution of a feature to an object by linking it to the wider context of those (other) features that are consequences of the given attribution. Lastly, the notion of *omniscience*, stipulating that the agent knows everything that is the case, is classically captured by the axiom $p \to \Box p$ corresponding on Kripke frames to the first-order condition $R \subseteq \Delta$. On polarity-based structures $\mathbb{F}$, the $\mathbf{L}_\Box$-sequent $\varphi \vdash \Box\varphi$ (cf. [4, Proposition 4.3]) corresponds to the first-order condition $I \subseteq R_\Box$, indicating that, whenever an object has a feature, the agent knows this.

Exercise 4. *Prove that for every polarity-based frame $\mathbb{F} = (\mathbb{P}, R_\Box)$,*

$$\mathbb{F} \models p \vdash \Box p \quad \textit{iff} \quad \forall a \forall x (aIx \Rightarrow aR_\Box x).$$

Hint: follow a similar strategy as Exercise 1.

Typicality formalized via 'common knowledge' In Lecture 1, we discussed that one of the desiderata of the logical theory of categorization is to be able to integrate seemingly dichotomous views of what categories are and do. One of the most important such dichotomous views is the one between the classical and the prototype view on categories. We proposed that the reconciliation should be effected by endowing the FCA-based framework with extra tools for encoding a notion of typicality.

The main insight guiding our proposal is that typicality is the outcome of an *intersubjective* process, in which agents do not only consider what they themselves would think as a typical member of a certain category, but would also take into account which objects, in their opinion, other agents would think of as typical members, and so on. Therefore, formalizing typicality requires an explicit formalization of this intersubjective process.

To this effect, in [5, 6], an expansion $\mathscr{L}_C$ of $\mathscr{L}$ was introduced with a common knowledge-type operator C. Given a denumerable set of proposition variables Prop and a finite set Ag of agents (the elements of which are $i \in \mathsf{Ag}$), the language $\mathscr{L}_C$ of the epistemic logic of categories with 'common knowledge' is:

$$\varphi := \bot \mid \top \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box_i \varphi \mid C(\varphi).$$

C-formulas are interpreted in models as follows:

$$\begin{array}{lll} \mathbb{M}, a \Vdash C(\varphi) & \text{iff} & \text{for all } x \in X, \text{ if } \mathbb{M}, x \succ \varphi, \text{ then } aR_C x \\ \mathbb{M}, x \succ C(\varphi) & \text{iff} & \text{for all } a \in A, \text{ if } \mathbb{M}, a \Vdash C(\varphi), \text{ then } aIx, \end{array}$$

where $R_C \subseteq A \times X$ is defined as $R_C = \bigcap_{s \in S} R_s$, and $R_s \subseteq A \times X$ is the relation associated with the modal operator $\Box_s := \Box_{i_1} \cdots \Box_{i_n}$ for any element $s = i_1 \cdots i_n$ in the set S of finite sequences of elements of Ag.

The basic logic of categories with 'common knowledge' is a set $\mathbf{L}_C$ of sequents $\varphi \vdash \psi$, with $\varphi, \psi \in \mathscr{L}_C$, which contains the axioms and is closed under the rules of $\mathbf{L}$, and in addition contains the following axioms:

$$\top \vdash C(\top) \qquad C(p) \wedge C(q) \vdash C(p \wedge q) \qquad C(p) \vdash \bigwedge \{\Box_i p \wedge \Box_i C(p) \mid i \in \mathsf{Ag}\}$$

and is closed under the following inference rules:

$$\frac{\varphi \vdash \psi}{C(\varphi) \vdash C(\psi)} \qquad \frac{\chi \vdash \bigwedge_{i \in \mathsf{Ag}} \Box_i \varphi \quad \{\chi \vdash \Box_i \chi \mid i \in \mathsf{Ag}\}}{\chi \vdash C(\varphi)}$$

The interpretation of C-formulas on models indicates that, for every category φ, the members of $C(\varphi)$ are those objects which are members of φ according to every agent, and moreover, according to every agent, are attributed membership in φ by every (other) agent, and so on. This provides justification for our proposal to regard the members of $C(\varphi)$ as the *(proto)typical members* of φ. The main feature of this proposal is that it is explicitly based on the agents' viewpoints. This feature is compatible with empirical methodologies adopted to establish graded membership (cf. [**?**]). Notice that there is a hierarchy of reasons why a given object fails to be a typical member of φ, the most severe being that some agents do not recognize its membership in φ, followed by some agents not recognizing that any other agent would recognize it as a member of φ, and so on. This observation provides a purely qualitative route to encode the *gradedness* of (the recognition of) category-membership (e.g. represented in a 'hybrid' language with (co-)nominal variables for designated objects **a**, **b** and features **x**). That is, two non-typical objects **a** and **b** can be compared in terms of the minimum number of 'epistemic iterations' needed for their typicality test to fail, so that **b** is *more atypical* than **a** if fewer rounds are needed for **b** than for **a**. This definition can be readily adapted so as to say that **b** is a *more atypical* member of ψ than **a** is of φ.

Lattice-based normal modal logic as the logic of rough concepts. As discussed above, the interpretation of $\mathbf{L}_\Box$ as an epistemic logic of formal concepts, facilitated by the polarity-based semantics, extends coherently from the meaning of the defining clauses of $\Vdash$ and $\succ$ relative to $\Box$-formulas, all the way to the preservation of the meaning of well known epistemic principles. However, the epistemic interpretation is not the only possible one; in what follows, we give pointers to another family of possible interpretations, proposed in [4], where polarity-based $\mathscr{L}$-frames are used to generalize Rough Set Theory (RST) [11] to the setting of *rough concepts*. The basic models in RST are pairs (X, R), called *approximation spaces*, with X a non-empty set and R an equivalence relation on X. The set X is to be thought of as the *domain of discourse* and R as an *indiscernibility relation*. The equivalence classes of R establish the granularity of the discourse by setting the limits to the distinctions that can be drawn. This granularity is captured algebraically by the upper and lower approximation operators arising from approximation spaces, which, when applied to any given subset $T \subseteq X$, encode the available information about T as follows. The lower approximation of $\underline{T}$ of T consists of those elements whose R-equivalence classes are contained in T, while the upper approximation of T consists of those elements whose R-equivalence classes have non-empty intersection with T. In other words,

$$\underline{T} := \bigcup\{R[z] \mid z \in T \text{ and } R[z] \subseteq T\} \quad \text{ and } \quad \overline{T} := \bigcup\{R[z] \mid z \in T\}.$$

The lower approximation $\underline{T}$ can be thought of as the set of all objects that are *definitely* in T, while the upper approximation $\overline{T}$ consists of those objects that are *possibly* in T.

As the reader would have remarked, an approximation space is nothing but a frame for the modal logic S5, and the lower and upper approximation of $T \subseteq S$ are obtained by applying the interior and closure operators given by the S5 box and diamond operators associated with the indiscernibility relation R, respectively. This connection with modal logic has indeed not gone unnoticed in the literature and has been elaborated in e.g. [9], [2] and [10].

In [4], *conceptual approximation spaces* were defined as polarity-based $\mathcal{L}$-frames $\mathbb{F} = (\mathbb{P}, R_\Box, R_\Diamond)$ such that $\mathbb{P} = (A, X, I)$ is a polarity, and $R_\Box \subseteq A \times X$ and $R_\Diamond \subseteq X \times A$ are I-compatible relations verifying the first-order conditions corresponding to the following modal axioms: $\Box\varphi \vdash \Diamond\varphi$ (*seriality*); $\Box\varphi \vdash \varphi$ and $\varphi \vdash \Diamond\varphi$ (*reflexivity*); $\Box\varphi \vdash \Box\Box\varphi$ and $\Diamond\Diamond\varphi \vdash \Diamond\varphi$ (*transitivity*); $\varphi \vdash \Box\Diamond\varphi$ and $\Diamond\Box\varphi \vdash \varphi$ (*symmetry*).

Taken together, these conditions guarantee that $\mathbb{F}^+ := (\mathbb{P}, [R_\Box], \langle R_\Diamond \rangle)$ is a complete lattice-based algebra such that $[R_\Box]$ and $\langle R_\Diamond \rangle$ are an interior and a closure operator respectively; moreover, $\langle R_\Diamond \rangle$ is the left adjoint of $[R_\Box]$ (i.e. $aR_\Box x$ iff $xR_\Diamond a$ for every $a \in A$ and $x \in X$).

Under the usual interpretation of $\mathbb{P} = (A, X, I)$ as a database, one possible way to understand $aR_\Box x$ or equivalently $xR_\Diamond a$ is 'there is *evidence* that object a has feature x', or 'object a *demonstrably* has feature x' (cf. [4, Section 5.1]). This intuitive understanding makes it plausible to assume that $R_\Box \subseteq I$. Recall that $\Vdash$ for $\Box$-formulas and of $\succ$ for $\Diamond$-formulas are defined as follows:

$$
\begin{array}{lll}
a \Vdash \Box\varphi & \text{iff} & \text{for all } x \in X, \text{ if } x \succ \varphi \text{ then } aR_\Box x \\
x \succ \Diamond\varphi & \text{iff} & \text{for all } a \in A, \text{ if } a \Vdash \varphi \text{ then } xR_\Diamond a.
\end{array}
$$

Under the interpretation discussed above, these clauses can be understood as saying that $\Box\varphi$ is the concept the examples/members of which are exactly those objects that *demonstrably* have all the features shared by φ-objects, and that $\Diamond\varphi$ is the concept described by the features which all φ-objects *demonstrably* have. Hence, $\Box\varphi$ can be understood as the (sub)concept of the *certified members* of φ, while $\Diamond\varphi$ as the (super) concept of the *potential members* of φ.

Thus, under the interpretation of $R_\Box$ and $R_\Diamond$ proposed above, the polarity-based semantics of $\mathcal{L}$ supports the understanding of $\Box\varphi$ and $\Diamond\varphi$ as the lower and upper approximations of concept φ, respectively. Notice that, while in approximation spaces the relation R relates indiscernible states, and thus directly encodes the extent of our *ignorance*, in the setting of conceptual approximation spaces, $R_\Box$ (or equivalently $R_\Diamond$) directly encode the (possibly partial) extent of our *knowledge* or information.

Exercise 5. *Prove that, for any enriched formal context* $\mathbb{F} = (\mathbb{P}, R_\Box, R_\Diamond)$*:*

1. $\mathbb{F} \models \Box\varphi \vdash \Diamond\varphi \quad$ *iff* $\quad R_\Box ; R_\blacksquare \subseteq I$.
2. $\mathbb{F} \models \Box\varphi \vdash \varphi \quad$ *iff* $\quad R_\Box \subseteq I$.
3. $\mathbb{F} \models \varphi \vdash \Diamond\varphi \quad$ *iff* $\quad R_\blacksquare \subseteq I$.
4. $\mathbb{F} \models \Box\varphi \vdash \Box\Box\varphi \quad$ *iff* $\quad R_\Box \subseteq R_\Box ; R_\Box$.
5. $\mathbb{F} \models \Diamond\Diamond\varphi \vdash \Diamond\varphi \quad$ *iff* $\quad R_\Diamond \subseteq R_\Diamond ; R_\Diamond$.

6. $\mathbb{F} \models \varphi \vdash \Box\Diamond\varphi$ *iff* $R_\Diamond \subseteq R_\blacklozenge$.

7. $\mathbb{F} \models \Diamond\Box\varphi \vdash \varphi$ *iff* $R_\blacklozenge \subseteq R_\Diamond$.

8. $\mathbb{F} \models \varphi \vdash \Box\varphi$ *iff* $I \subseteq R_\Box$.

9. $\mathbb{F} \models \Diamond\varphi \vdash \varphi$ *iff* $I \subseteq R_\blacksquare$.

10. $\mathbb{F} \models \Box\Box\varphi \vdash \Box\varphi$ *iff* $R_\Box ; R_\Box \subseteq R_\Box$.

11. $\mathbb{F} \models \Diamond\varphi \vdash \Diamond\Diamond\varphi$ *iff* $R_\Diamond ; R_\Diamond \subseteq R_\Diamond$.

12. $\mathbb{F} \models \Diamond\varphi \vdash \Box\varphi$ *iff* $I \subseteq R_\blacksquare ; R_\Box$.

From concepts to other ontologies. In [4], other more specific interpretations are proposed concerning situations which span from the analysis of text databases to medical diagnoses and the analysis of markets. Accordingly, in each of these situations, $\Box\varphi$ and $\Diamond\varphi$ can be given more specific interpretations as lower and upper approximations of concepts or categories or relevant clusters.

For instance (cf. [4, Section 5.4,] modified), text databases can be modelled as polarity-based $\mathscr{L}_{\mathbf{PML}}$-structures $\mathbb{F} = (\mathbb{P}, R_\Box, R_\Diamond)$ such that $\mathbb{P} = (A, X, I)$ with A being a set of documents, X a set of words, and aIx being understood as 'document a has word x as a keyword'. Formal concepts arising from such an $\mathbb{F}$ can be understood as *themes* or *topics*, intensionally described by Galois-stable sets of (key)words. In this situation, one of the many possible interpretations of $aR_\Box x$ or equivalently $xR_\Diamond a$ is 'document a has word x as its *first or second* keyword', which again makes it plausible to assume that $R_\Box \subseteq I$.

As another example (cf. [4, Section 5.5] modified), let $\mathbb{P} = (A, X, I)$ represent a hospital, where A is the set of patients, X is the set of symptoms, and aIx iff "patient a has symptom x". Concepts arising from this representation are *syndromes*, intensionally described by Galois-stable sets of symptoms. In this situation, let $aR_\Box x$, or equivalently $xR_\Diamond a$, iff 'a has been *tested* for symptom x with positive outcome'.

As a third example (cf. [4, Section 5.8] modified), let $\mathbb{P} = (A, X, I)$ where A is the set of consumers, X is the set of market-products, and aIx iff 'consumer a buys product x'. Concepts arising from this representation are *consumer segments*, intensionally described by Galois-stable sets of market-products. In this situation, let $aR_\Box x$, or equivalently $xR_\Diamond a$, iff 'a buys x from a certain producer i'. Then $\Box\varphi$ denotes the market share of producer i in consumer segment φ.

As a fourth example, let $\mathbb{P} = (A, X, I)$ where A is the set of empirical hypotheses, X is the set of variables, and aIx iff 'hypothesis a is formulated in terms of variable x'. Concepts arising from this representation are empirical *theories*, extensionally described by Galois-stable sets of hypotheses and intensionally described by Galois-stable sets of variables. In this situation, let $aR_\Box x$, or equivalently $xR_\Diamond a$, iff 'x is a *dependent* variable for hypothesis a'. Then, if $\mathbf{x} = (x^{\downarrow}, x^{\downarrow\uparrow})$ denotes the formal concept generated by x, the extension of $\Box\mathbf{x}$ contains all hypotheses that compete with each other.

Finally, let $\mathbb{P} = (A, X, I)$ represent a decision-making situation in which A is the set of decision-makers, X is the set of issues, and aIx iff 'agent a finds issue x relevant'.

Concepts arising from this representation are *interrogative agendas*, extensionally described by Galois-stable *coalitions* and intensionally described by Galois-stable sets of issues. In this situation, let $aR_{\Box}x$, or equivalently $xR_{\Diamond}a$, iff 'agent a regards x as a positive issue'. For example, if A is the set of the members of a hiring committee and X is the set of the features of potential applicants, agent a could regard "the candidate obtained their PhD recently" as a desirable characteristic, i.e. a positive issue, while other agents might prefer a more experienced candidate and therefore not regard this as positive. This would mean that $R_{\Box} \subseteq I$ and that, extensionally, $\Box\varphi$ would be the coalition of all agents who are positive towards all issues on interrogative agenda φ.

References

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