Logical foundations of categorization theory
Lecture 3

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Abstract

In this lecture, we discuss how formal contexts can be seen as (sound and complete) relational models of a basic propositional logic of categories and concepts.

1 Polarities as models for the basic lattice logic

The discussion in the previous classes justifies the proposal that complete lattices are the most fundamental structures in which categories and concepts can be formalized. In this section, we start by introducing the most basic logic naturally arising from complete lattices.

Basic logic and informal understanding. Let Prop be a (countable or finite) set of atomic category-labels. The language $\mathscr{L}$ of the *basic propositional logic of formal concepts* is

$$\varphi := \bot \mid \top \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi$$

where $p \in \mathsf{Prop}$. Clearly, the logical signature of this language matches the algebraic signature of the complex algebra of any formal context $\mathbb{P}$. Hence, formulas in this language can be interpreted as formal concepts of $\mathbb{P}$. If the formal context $\mathbb{P}$ is regarded as the abstract representation of a database, atomic propositions $p \in \mathsf{Prop}$ can be understood as *atomic labels* (or *names*) for concepts, appropriate to the nature of the database. For instance, if the database consists of music albums and their features (e.g. names of performers, types of musical instruments, number of bits per minute etc), then the atomic propositions can stand for names of music genres (e.g. jazz, rock, rap); likewise, if the database consists of movies and their features (e.g. names of directors or performers, duration, presence of special effects, presence of costumes, presence of shooting scenes, etc), then the atomic propositions can stand for movie genres (e.g. western, drama, horror); if the database consists of goods on sale in a supermarket and their features (e.g. capacity of packages, presence of additives, presence of organic certification, etc) then the conceptual labels can stand for supermarket categories (e.g. detergents, dairies, spices); if the database consists of the patients in a hospital and their symptoms (e.g. fever, jaundice, vertigos, etc), then the atomic propositions can stand for diseases (e.g. pneumonia, hepatitis, diabetes). Compound formulas $\varphi \wedge \psi$ and $\varphi \vee \psi$ respectively denote the greatest common subconcept and the smallest common superconcept of φ and ψ. The *basic*, or *minimal* $\mathcal{L}$*-logic* is a set $\mathbf{L}$ of sequents $\varphi \vdash \psi$ (which intuitively read "φ is a subconcept of ψ") with $\varphi, \psi \in \mathcal{L}$, containing the following axioms:

- Sequents for propositional connectives:

$$
\begin{array}{llll}
p \vdash p, & \bot \vdash p, & p \vdash \top, & \\
p \vdash p \vee q, & q \vdash p \vee q, & p \wedge q \vdash p, & p \wedge q \vdash q,
\end{array}
$$

and closed under the following inference rules:

$$
\frac{\varphi \vdash \chi \quad \chi \vdash \psi}{\varphi \vdash \psi} \qquad \frac{\varphi \vdash \psi}{\varphi(\chi/p) \vdash \psi(\chi/p)} \qquad \frac{\chi \vdash \varphi \quad \chi \vdash \psi}{\chi \vdash \varphi \wedge \psi} \qquad \frac{\varphi \vdash \chi \quad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi}
$$

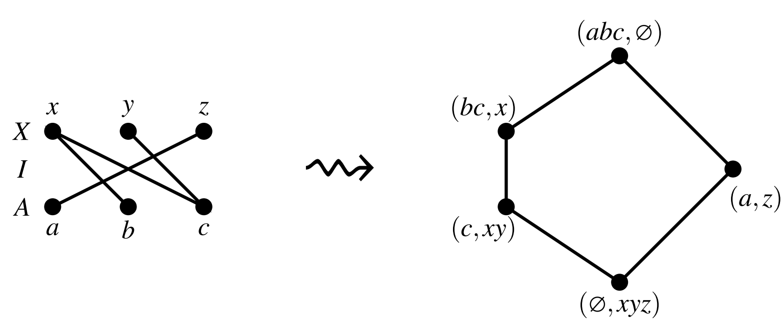
By an $\mathcal{L}$*-logic* we understand any extension of $\mathbf{L}$ with $\mathcal{L}$-axioms $\varphi \vdash \psi$.

Interpretation in formal contexts. For any polarity $\mathbb{P} = (A, X, I)$ a *valuation* on $\mathbb{P}$ is a map $V : \mathsf{Prop} \to \mathbb{P}^+$. An $\mathcal{L}$*-model* is a tuple $\mathbb{M} = (\mathbb{P}, V)$. For every atomic category label $p \in \mathsf{Prop}$, we let $[\![p]\!] := [\![V(p)]\!]$ (resp. $(\![p]\!) := (\![V(p)]\!)$) denote the extension (resp. the intension) of the interpretation of p in $\mathbb{M}$. The elements of $[\![p]\!]$ are the *members* of concept p in $\mathbb{M}$; the elements of $(\![p]\!)$ *describe* concept p in $\mathbb{M}$. Alternatively, we write:

$$
\begin{array}{lll}
\mathbb{M}, a \Vdash p & \text{iff} & a \in [\![p]\!]_{\mathbb{M}} \\
\mathbb{M}, x \succ p & \text{iff} & x \in (\![p]\!)_{\mathbb{M}}
\end{array}
$$

and we read $\mathbb{M}, a \Vdash p$ as "object a is a member of category p", and $\mathbb{M}, x \succ p$ as "feature x describes category p".

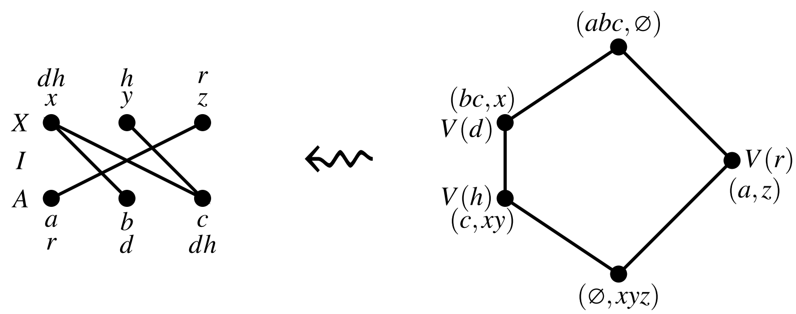
Let us illustrate this definition with a concrete example. Consider the polarity $\mathbb{P} = (A, X, I)$, representing a 'database' of theatrical plays the set of objects of which is $A := \{a, b, c\}$, where a is *A Midsummer Night's Dream*, b is *King Lear*, and c is *Julius Caesar*, while its set of features is $X := \{x, y, z\}$, where x is *'no happy end'*, y is *'some characters are real historical figures'*, and z is *'two characters fall in love with each other'*. The following picture represents $\mathbb{P}$ and its associated concept lattice $\mathbb{P}^+$.



Let $\mathsf{Prop} := \{r, d, h\}$ be the set of atomic concept-variables, where r stands for 'romantic comedy', d for 'drama' and h for 'historical drama'. Consider the assignment $V : \mathsf{Prop} \to \mathbb{P}^+$ which maps r to (a, z), d to (bc, x) and h to (c, xy). Then, for every $p \in \mathsf{Prop} := \{r, d, h\}$, the clauses

$$\begin{array}{lll} \mathbb{M}, a \Vdash p & \text{iff} & a \in [\![p]\!]_{\mathbb{M}} \\ \mathbb{M}, x \succ p & \text{iff} & x \in (\![p]\!)_{\mathbb{M}} \end{array}$$

instantiate as shown in the picture below:



Interpreting arbitrary formulas. As usual, the valuation V can be homomorphically extended to an interpretation map of $\mathscr{L}$-formulas, also denoted V, defined as follows:

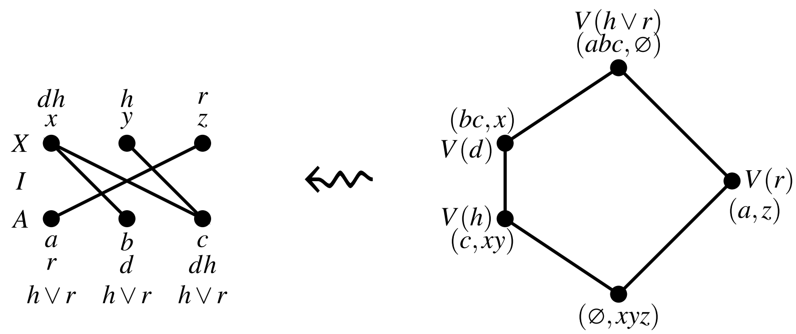
$$\begin{array}{rcl} V(p) & = & ([\![p]\!], (\![p]\!)) \\ V(\top) & = & (A, A^{\uparrow}) \\ V(\bot) & = & (X^{\downarrow}, X) \\ V(\varphi \wedge \psi) & = & ([\![\varphi]\!] \cap [\![\psi]\!], ([\![\varphi]\!] \cap [\![\psi]\!])^{\uparrow}) \\ V(\varphi \vee \psi) & = & (((\![\varphi]\!) \cap (\![\psi]\!))^{\downarrow}, (\![\varphi]\!) \cap (\![\psi]\!)) \end{array}$$

Hence, in each model, $\top$ is interpreted as the concept generated by the set A of all objects, i.e. the widest concept and hence the one with the laxest (possibly empty) description; $\bot$ is interpreted as the category generated by the set X of all features, i.e. the smallest category and hence the one with the most restrictive description and possibly empty extension; $\varphi \wedge \psi$ is interpreted as the semantic category determined by the intersection of the extensions of φ and ψ (hence, the description of $\varphi \wedge \psi$ certainly includes $(\![\varphi]\!) \cup (\![\psi]\!)$ but can be strictly larger, as shown below). Likewise, $\varphi \vee \psi$ is interpreted as the semantic category determined by the intersection of the intensions of φ and ψ (hence, it is always the case that $[\![\varphi]\!] \cup [\![\psi]\!] \subseteq [\![\varphi \vee \psi]\!]$ but this inclusion can be strict, as we will show below).

The homomorphic extension of each valuation gives rise to the recursive definition of the relations of "membership" $\Vdash$ of objects in categories, and of features "describing" categories ($\succ$) extended to all $\mathscr{L}$-formulas. Before giving the formal definition, let us illustrate this definition in the context of our example. Since the assignment V maps r to (a, z), and h to (c, xy), we can compute the interpretation of $h \vee r$ induced by V as follows:

$$V(h \vee r) = V(h) \vee V(r) = (a, z) \vee (c, xy) = (abc, \varnothing).$$

This translates into the possibility of extending the "membership" and "description" relations $\Vdash$ and $\succ$ to $h \vee r$ as illustrated in the following picture:



Notice that $\mathbb{M}, b \Vdash h \vee r$, however, $\mathbb{M}, b \nVdash h$ and $\mathbb{M}, b \nVdash r$. Likewise, reasoning analogously, one shows that $V(d \wedge r) = (\varnothing, xyz)$, and hence $\mathbb{M}, y \succ d \wedge r$, however, $\mathbb{M}, y \not\succ d$ and $\mathbb{M}, y \not\succ r$.

In general, spelling out the definition of the homomorphic extension of a given assignment on the concept lattice of a polarity according to the following conditions:

$$\begin{array}{lll} \mathbb{M}, a \Vdash \varphi & \text{iff} & a \in [\![\varphi]\!]_{\mathbb{M}} \\ \mathbb{M}, x \succ \varphi & \text{iff} & x \in (\![\varphi]\!)_{\mathbb{M}} \end{array}$$

yields the following recursive definition of the "membership relation" $\Vdash$ of objects in categories, and of features "describing" categories ($\succ$) extended to all $\mathscr{L}$-formulas:

$$\begin{array}{lll}
\mathbb{M}, a \Vdash \top & & \text{always} \\
\mathbb{M}, x \succ \top & \text{iff} & aIx \text{ for all } a \in A \\
\mathbb{M}, x \succ \bot & & \text{always} \\
\mathbb{M}, a \Vdash \bot & \text{iff} & aIx \text{ for all } x \in X \\
\mathbb{M}, a \Vdash \varphi \wedge \psi & \text{iff} & \mathbb{M}, a \Vdash \varphi \text{ and } \mathbb{M}, a \Vdash \psi \\
\mathbb{M}, x \succ \varphi \wedge \psi & \text{iff} & \text{for all } a \in A, \text{ if } \mathbb{M}, a \Vdash \varphi \wedge \psi, \text{ then } aIx \\
\mathbb{M}, x \succ \varphi \vee \psi & \text{iff} & \mathbb{M}, x \succ \varphi \text{ and } \mathbb{M}, x \succ \psi \\
\mathbb{M}, a \Vdash \varphi \vee \psi & \text{iff} & \text{for all } x \in X, \text{ if } \mathbb{M}, x \succ \varphi \vee \psi, \text{ then } aIx
\end{array}$$

Finally, as to the interpretation of sequents:

$$\begin{array}{lllll}
\mathbb{M} \models \varphi \vdash \psi & \text{iff} & [\![\varphi]\!] \subseteq [\![\psi]\!] & \text{iff} & \text{for all } a \in A, \text{ if } \mathbb{M}, a \Vdash \varphi, \text{ then } \mathbb{M}, a \Vdash \psi \\
& \text{iff} & (\![\psi]\!) \subseteq (\![\varphi]\!) & \text{iff} & \text{for all } x \in X, \text{ if } \mathbb{M}, x \succ \psi, \text{ then } \mathbb{M}, x \succ \varphi.
\end{array}$$

A sequent $\varphi \vdash \psi$ is *valid* in a formal context $\mathbb{P}$ (in symbols: $\mathbb{P} \models \varphi \vdash \psi$) if $\mathbb{M} \models \varphi \vdash \psi$ for every model $\mathbb{M}$ based on $\mathbb{P}$.

Exercise 1 (Soundess)**.** *Verify that the axioms and rules of the basic logic of formal concepts are valid in any formal context. Conclude that the basic logic of formal concepts is sound w.r.t. the class of formal contexts.*

Exercise 2 (Failure of distributive laws)**.** *Verify that, if $\mathbb{M} = (\mathbb{P}, V)$ is the model described above, then*

$$\mathbb{M} \not\models d \wedge (h \vee r) \vdash (d \wedge h) \vee (d \wedge r) \quad \textit{and} \quad \mathbb{M} \not\models (h \vee d) \wedge (h \vee r) \vdash h \vee (r \wedge d).$$

2 Completeness

The completeness of **L** can be proven via a standard canonical model construction. For any lattice $\mathbb{L}$, let $\mathbb{P}_\mathbb{L} := (A, X, I)$ where A (resp. X) is the set of lattice filters (resp. ideals) of $\mathbb{L}$, and aIx iff $a \cap x \neq \varnothing$. The *canonical formal context* is defined by instantiating the construction above to the Lindenbaum-Tarski algebra of **L** (which we also denote **L**). In this case, let V be the *canonical valuation*, i.e. the one such that $[\![p]\!]$ (resp. $(\![\mathrm{p}]\!)$) is the set of the filters (resp. ideals) of **L** to which p belongs, and let $\mathbb{M} = (\mathbb{P}_\mathbf{L}, V)$ be the canonical model.

Exercise 3. *Verify that if* $[\![p]\!] := \{a \in A \mid p \in a\}$ *and* $([p]) := \{x \in X \mid p \in x\}$ *for every* $p \in \mathsf{Prop}$*, then* $[\![p]\!]^\uparrow = ([p])$ *and* $([p])^\downarrow = [\![p]\!]$*. Deduce that* V *is well defined.*

Then the following holds for $\mathbb{M}$:

Lemma 1 (Truth lemma)**.** *For every* $\varphi \in \mathscr{L}$*,*

1. $\mathbb{M}, a \Vdash \varphi$ *iff* $\varphi \in a$*;*
2. $\mathbb{M}, x \succ \varphi$ *iff* $\varphi \in x$*.*

Proof. By induction on φ. If $\varphi := p \in \mathsf{Prop}$ the statement follows by the definition of canonical valuation.

If $\varphi := \bot$ then $\bot \in x$ for any ideal x, and moreover, by definition of $\succ$, if $\mathbb{M}$ is any (polarity-based) model, $\mathbb{M}, x \succ \bot$ for any $x \in X$, so the required equivalence is verified. By definition, $\mathbb{M}$ is any (polarity-based) model, $\mathbb{M}, a \Vdash \bot$ iff aIx for any $x \in X$. By the definition of I in $\mathbb{P}_\mathbf{L}$, this is equivalent to stating that the filter a has nonempty intersection with every ideal of **L**, or equivalently, with the smallest of them, i.e. $a \cap \{\bot\} \neq \varnothing$, iff $\bot \in a$, as required.

As to the inductive step for $\varphi := \sigma \vee \xi$,

	$\mathbb{M}, x \succ \sigma \vee \xi$	
iff	$\mathbb{M}, x \succ \sigma$ and $\mathbb{M}, x \succ \xi$	definition of $\succ$ for $\vee$-formulas
iff	$\sigma \in x$ and $\xi \in x$	induction hypothesis
iff	$\sigma \vee \xi \in x$	x is an ideal

	$\mathbb{M}, a \Vdash \sigma \vee \xi$	
iff	aIx for any $x \in X$ s.t. $\mathbb{M}, x \succ \sigma \vee \xi$	definition of $\Vdash$ for $\vee$-formulas
iff	$a \cap x \neq \varnothing$ for any ideal x s.t. $\sigma \vee \xi \in x$	proof above
iff	$a \cap (\sigma \vee \xi)\downarrow \neq \varnothing$	$(\sigma \vee \xi)\downarrow$ is the smallest ideal x s.t. $\sigma \vee \xi \in x$
iff	$\sigma \vee \xi \in a$.	

The remaining cases ($\varphi := \top$ and $\varphi := \sigma \wedge \xi$) are left to the reader. ◻

Exercise 4. *Complete the proof of the truth lemma.*

Proposition 1 (Completeness)**.** *If* $\varphi \vdash \psi$ *is an* $\mathscr{L}$*-sequent which is not derivable in* **L***, then* $\mathbb{M} \not\models \varphi \vdash \psi$*.*

Proof. If the $\mathscr{L}$-sequent $\varphi \vdash \psi$ is not derivable in $\mathbf{L}$, then $a \cap x = \varnothing$, where a denotes the filter in the Lindenbaum-Tarski algebra generated by φ and x denotes the ideal in the Lindenbaum-Tarski algebra generated by ψ. Then the Truth Lemma implies that $a \in [\![\varphi]\!]$ and $a \notin [\![\psi]\!]$, hence $[\![\varphi]\!] \nsubseteq [\![\psi]\!]$, i.e. $\mathbb{M} \not\models \varphi \vdash \psi$, as required. $\square$