Logical foundations of categorization theory
Handout 5

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1 Composition of relations on polarities

In what follows, we fix two sets A and X, and use a, b (resp. x, y) for elements of A (resp. X), and B, C, A_j (resp. Y, W, X_j) for subsets of A (resp. of X) throughout this section. For any relation $S \subseteq U \times W$, let

$$S^{(1)}[U'] := \{w \mid \forall u(u \in U' \Rightarrow uSw)\} \qquad S^{(0)}[W'] := \{u \mid \forall w(w \in W' \Rightarrow uSw)\}.$$

Let us recall that the standard composition of $R, T \subseteq S \times S$ is

$$R \circ T := \{(u, w) \mid (u, v) \in R \text{ and } (v, w) \in T \text{ for some } v \in S\}.$$

Definition 1. *For any formal context $\mathbb{P} = (A, X, I)$,*

1. *for all relations $R, T \subseteq X \times A$, the* I-composition *$R\,;_I T \subseteq X \times A$ is such that, for any $a \in A$ and $x \in X$,*

$$(R\,;_I T)^{(0)}[a] = R^{(0)}[I^{(0)}[T^{(0)}[a]]], \quad \textit{i.e.} \quad x(R\,;_I T)a \quad \textit{iff} \qquad x \in R^{(0)}[I^{(0)}[T^{(0)}[a]];$$

2. *for all relations $R, T \subseteq A \times X$, the* I-composition *$R\,;_I T \subseteq A \times X$ is such that, for any $a \in A$ and $x \in X$,*

$$(R\,;_I T)^{(0)}[x] = R^{(0)}[I^{(1)}[T^{(0)}[x]], \quad \textit{i.e.} \quad a(R\,;_I T)x \quad \textit{iff} \qquad a \in R^{(0)}[I^{(1)}[T^{(0)}[x]].$$

When the context is fixed and clear, we will simplify notation and write e.g. $R;T$ in stead of $R;_I T$. In these cases, we will also refer to I-composition as 'composition'.

Exercise 1. *If $\mathbb{P} = (A, X, I)$ is a formal context, then*

1. *for any I-compatible relations $R, T \subseteq X \times A$ and any $a \in A$ and $x \in X$,*

$$x(R\,;T)a \quad \textit{iff} \quad (R\,;T)^{(1)}[x] = T^{(1)}[I^{(1)}[R^{(1)}[x]]];$$

2. *for any I-compatible relations $R, T \subseteq X \times A$ and any $a \in A$ and $x \in X$,*

$$a(R\,;T)x \quad \textit{iff} \quad (R\,;T)^{(1)}[a] = T^{(1)}[I^{(0)}[R^{(1)}[a]]].$$

Exercise 2. *Let $\mathbb{P} = (A, X, I)$ be a formal context. If $R, T \subseteq X \times A$ (resp. $R, T \subseteq A \times X$) and R is I-compatible, then so is $R\,;T$.*

Exercise 3. *Let $\mathbb{P} = (A, X, I)$ be a formal context.*

1. *If $R, T, U \subseteq X \times A$ are I-compatible, then $(R\,;T)\,;U = R\,;(T\,;U)$.*
2. *If $R, T, U \subseteq A \times X$ are I-compatible, then $(R\,;T)\,;U = R\,;(T\,;U)$;*

2 Lifting relations

Throughout this section, for every set S, we let $\Delta_S := \{(s,s) \mid s \in S\}$, and we typically drop the subscript when it does not cause ambiguities. Hence we write e.g. $\Delta^c = \{(s,s') \mid s, s' \in S \text{ and } s \neq s'\}$. We let S_A and S_X be copies of S, and for every $P \subseteq S$, we let $P_A \subseteq S_A$ and $P_X \subseteq S_X$ denote the corresponding copies of P in S_A and S_X, respectively. Then P^c_X (resp. P^c_A) stands both for $(P^c)_X$ (resp. $(P^c)_A$) and $(P_X)^c$ (resp. $(P_A)^c$). For the sake of a more manageable notation, we will use a and b (resp. x and y) to indicate both elements of A (resp. X) and their corresponding elements in S_A (resp. S_X), relying on the types of the relations for disambiguation.

Definition 2. *For every $R \subseteq S \times S$, we let*

1. *$I_R \subseteq S_A \times S_X$ such that aI_Rx iff aRx;*
2. *$J_R \subseteq S_X \times S_A$ such that xJ_Ra iff xRa.*

Exercise 4. *For any set S we let $\mathbb{P}_S := (S_A, S_X, I_{\Delta^c})$. Prove that $\mathbb{P}^+_S \cong \mathscr{P}(S)$. Hint: define $h : \mathscr{P}(S) \to \mathbb{P}^+_S$ as the assignment $P \mapsto (P_A, P^c_X)$.*

Exercise 5. *For all $R, T \subseteq S \times S$,*

1. *$I_{(R\circ T)^c} = I_{R^c}\,;I_{T^c}$.*
2. *$J_{(R\circ T)^c} = J_{R^c}\,;J_{T^c}$.*

Exercise 6. *For any Kripke frame $\mathbb{X} = (S, R)$, we let $\mathbb{F}_{\mathbb{X}} := (\mathbb{P}_S, I_{R^c}, J_{R^c})$ where $\mathbb{P}_S = (S_A, S_X, I_{\Delta^c})$ is defined as above. Show that $\mathbb{F}_{\mathbb{X}}$ is an enriched formal context. Hint: the relations $I_{R^c} \subseteq S_A \times S_X$ and $J_{R^c} \subseteq S_X \times S_A$ are trivially I_{Δ^c}-compatible.*

Definition 3. *A relation $R \subseteq S \times S$ is* sub-delta *if $R = \{(z,z) \mid z \in Z\}$ for some $Z \subseteq S$, and is* dense *if $\forall s \forall t[sRt \Rightarrow \exists u(sRu \;\&\; uRt)]$.*

Exercise 7. *For any Kripke frame $\mathbb{X} = (S, R)$,*

1. *R is reflexive iff $I_{R^c} \subseteq I_{\Delta^c}$ iff $J^{-1}_{R^c} \subseteq I_{\Delta^c}$.*
2. *R is transitive iff $I_{R^c} \subseteq I_{R^c}\,;I_{R^c}$ iff $J_{R^c} \subseteq J_{R^c}\,;J_{R^c}$.*
3. *R is symmetric iff $I_{R^c} = J^{-1}_{R^c}$ iff $J_{R^c} = I^{-1}_{R^c}$.*
4. *R is sub-delta iff $I_{\Delta^c} \subseteq I_{R^c}$ iff $I_{\Delta^c} \subseteq J^{-1}_{R^c}$.*

5. R is dense iff $I_{R^c}\,; I_{R^c} \subseteq I_{R^c}$ iff $J_{R^c}\,; J_{R^c} \subseteq J_{R^c}$.

Definition 4. *For any polarity $\mathbb{P} = (A, X, I)$, a relation $R \subseteq A \times X$ (resp. $T \subseteq X \times A$) is*

$$\begin{array}{rllll} \textit{reflexive} & \textit{iff} & R \subseteq I & (\textit{resp. iff} & T^{-1} \subseteq I) \\ \textit{transitive} & \textit{iff} & R \subseteq R\,; R & (\textit{resp. iff} & T \subseteq T\,; T) \\ \textit{subdelta} & \textit{iff} & I \subseteq R & (\textit{resp. iff} & I \subseteq T^{-1}) \\ \textit{dense} & \textit{iff} & R\,; R \subseteq R & (\textit{resp. iff} & T\,; T \subseteq T). \end{array}$$

3 Exercises from Lecture 5

Exercise 8. *Propose an informal understanding, along the lines in Section 2 of Lecture 5, of the following interpretation clauses:*

1. $\mathbb{M}, x \succ \Box\varphi$ iff for all $a \in A$, if $\mathbb{M}, a \Vdash \Box\varphi$, then aIx.
2. $\mathbb{M}, a \Vdash \Diamond\varphi$ iff for all $x \in X$, if $\mathbb{M}, x \succ \Diamond\varphi$, then aIx
3. $\mathbb{M}, x \succ \Diamond\varphi$ iff for all $a \in A$, if $\mathbb{M}, a \Vdash \varphi$, then $xR_\Diamond a$.

Exercise 9. *Prove that for every polarity-based frame $\mathbb{F} = (\mathbb{P}, R_\Box)$,*

$$\mathbb{F} \models \Box p \vdash p \quad \textit{iff} \quad \forall a \forall x (aR_\Box x \Rightarrow aIx).$$

Hint: from right to left, fix a valuation V on $\mathbb{P}^+$, and show that, for every $a \in A$, if $a \in [\![\Box p]\!]_V$ then $a \in [\![p]\!]_V$; for the converse direction, assume that $aR_\Box x$ but not aIx for some $a \in A$ and $x \in X$, and find an assignment V on $\mathbb{P}^+$ such that $\mathbb{F} \not\models \Box p \vdash p$.

Exercise 10. *Prove that for every polarity-based frame $\mathbb{F} = (\mathbb{P}, R_\Box)$,*

$$\mathbb{F} \models \Box p \vdash \Box\Box p \quad \textit{iff} \quad R_\Box \subseteq R_\Box \;;_I R_\Box.$$

Exercise 11. *Prove that for every polarity-based frame $\mathbb{F} = (\mathbb{P}, R_\Box)$,*

$$\mathbb{F} \models p \vdash \Box p \quad \textit{iff} \quad \forall a \forall x (aIx \Rightarrow aR_\Box x).$$

Hint: follow a similar strategy as Exercise 2.

Exercise 12. *Prove that, for any enriched formal context $\mathbb{F} = (\mathbb{P}, R_\Box, R_\Diamond)$:*

1. $\mathbb{F} \models \Box\varphi \vdash \Diamond\varphi$ iff $R_\Box ; R_\blacksquare \subseteq I$.
2. $\mathbb{F} \models \Box\varphi \vdash \varphi$ iff $R_\Box \subseteq I$.
3. $\mathbb{F} \models \varphi \vdash \Diamond\varphi$ iff $R_\blacksquare \subseteq I$.
4. $\mathbb{F} \models \Box\varphi \vdash \Box\Box\varphi$ iff $R_\Box \subseteq R_\Box ; R_\Box$.
5. $\mathbb{F} \models \Diamond\Diamond\varphi \vdash \Diamond\varphi$ iff $R_\Diamond \subseteq R_\Diamond ; R_\Diamond$.
6. $\mathbb{F} \models \varphi \vdash \Box\Diamond\varphi$ iff $R_\Diamond \subseteq R_\blacklozenge$.

7. $\mathbb{F} \models \Diamond\Box\varphi \vdash \varphi$ *iff* $R_{\blacklozenge} \subseteq R_{\Diamond}$.

8. $\mathbb{F} \models \varphi \vdash \Box\varphi$ *iff* $I \subseteq R_{\Box}$.

9. $\mathbb{F} \models \Diamond\varphi \vdash \varphi$ *iff* $I \subseteq R_{\blacksquare}$.

10. $\mathbb{F} \models \Box\Box\varphi \vdash \Box\varphi$ *iff* $R_{\Box} ; R_{\Box} \subseteq R_{\Box}$.

11. $\mathbb{F} \models \Diamond\varphi \vdash \Diamond\Diamond\varphi$ *iff* $R_{\Diamond} ; R_{\Diamond} \subseteq R_{\Diamond}$.

12. $\mathbb{F} \models \Diamond\varphi \vdash \Box\varphi$ *iff* $I \subseteq R_{\blacksquare} ; R_{\Box}$.