Logical foundations of categorization theory
Handout 4

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1 Preliminaries

Definition 1 (Adjunction)**.** *Let $P = (X, \leq)$ and $Q = (Y, \leq)$ be posets. The pair of maps $(f : X \to Y; g : Y \to X)$ is an* adjoint pair *if for every $x \in X$, $y \in Y$,*

$$f(x) \leq y \ \text{ iff } \ x \leq g(y).$$

In this case, f is the left *or* lower adjoint *and g is the* right*, or* upper adjoint*, and we write: $f \dashv g$.*

Exercise 1. *Let $P = (X, \leq)$ and $Q = (Y, \leq)$ be posets, and the pair of maps $(f : X \to Y; g : Y \to X)$ be an* adjoint pair*. Prove that, for any $x \in X$,*

1. *$fg(x) \leq x \leq gf(x)$;*
2. *f, g are monotone.*[1]

Definition 2. *For every $P = (X, \leq)$, $Q = (Y, \leq)$, a map $f : X \to Y$* preserves existing joins *if for every $S \subseteq X$, if $\bigvee S$ exists in P, then $\bigvee f[S]$ exists in Q and $f(\bigvee S) = \bigvee f[S]$. A map $g : X \to Y$* preserves existing meets *if for every $S \subseteq X$, if $\bigwedge S$ exists in P, then $\bigwedge g[S]$ exists in Q and $g(\bigwedge S) = \bigwedge g[S]$.*

Exercise 2. *Let $P = (X, \leq)$ and $Q = (Y, \leq)$ be posets, and let $f : X \to Y$ and $g : Y \to X$ be maps.*

1. *Prove that if $f \dashv g$ then then f preserves (existing) joins and g preserves (existing) meets.*
2. *Prove that if P is a complete lattice then the following are equivalent,*
   (a) *f preserves (existing) joins;*
   (b) *$f \dashv g$ for some $g : Y \to X$.*

[1] Let $P = (X, \leq)$ and $Q = (Y, \leq)$ be posets, a map $f : X \to Y$ is monotone if, for any $x, y \in X$, if $x \leq y$ then $f(x) \leq f(y)$.

Hint: for (a) implies (b), let $g(y) := \bigvee\{x \in X \mid f(x) \leq y\}$ *for any* $y \in Y$.

3. *Deduce from the item above that if Q is a complete lattice then the following are equivalent,*

   (a) *g preserves (existing) meets;*

   (b) $f \dashv g$ *for some* $f : X \to Y$.

Hint: for (a) implies (b), let $f(x) := \bigwedge\{y \in Y \mid x \leq g(y)\}$ *for any* $x \in X$.

2 Exercises from Lecture 4

Exercise 3. *Prove Lemma 1 of Lecture 4.*

Exercise 4. *Complete the proof of Lemma 3 of Lecture 4.*

Exercise 5. *Complete the proof of Lemma 4 of Lecture 4.*

Exercise 6. *Complete the proof of Lemma 5 of Lecture 4.*

Exercise 7. *Complete the proof of Proposition 2 of Lecture 4.*

Exercise 8. *Prove Lemma 6 of Lecture 4.*

Exercise 9. *Complete the proof of Lemma 7 of Lecture 4.*

Exercise 10. *Complete the proof of Lemma 8 of Lecture 4.*

Exercise 11. *Complete the proof of the truth lemma of Lecture 4.*