Logical foundations of categorization theory
Handout 3

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1 Ideals and filters

Definition 1 (Ideal and filter)**.** *Let L be a lattice. A non-empty set $J \subseteq L$ is called an* ideal *if*

(a) for all $a,b \in L$, if $a \in L$, $b \in J$ and $a \leq b$ then $a \in J$;

(b) for all $a,b \in L$, if $a,b \in J$ then $a \vee b \in J$.

Dually, a non-empty set $G \subseteq L$ is called a filter *if*

(a') for all $a,b \in L$, if $a \in L$, $b \in G$ and $b \leq a$ then $a \in G$;

(b') for all $a,b \in L$, if $a,b \in G$ then $a \wedge b \in G$.

An ideal or filter is called proper *if it does not coincide with L.*

Exercise 1. *Let (X,τ) be a topological space and let $x \in X$. Show that the set $\{V \subseteq X \mid (\exists U \in \tau) x \in U \subseteq V\}$ is a filter in $\mathscr{P}(X)$.*

Exercise 2. *Let L be a lattice. Show that the set $\downarrow a := \{x \in L \mid x \leq a\}$ is an ideal, and $\uparrow a := \{x \in L \mid a \leq x\}$ is a filter.*

Definition 2 (Generated ideal and filter)**.** *Let L be a lattice and $X \subseteq L$ be non-empty. The set $\lceil X \rceil := \{x \in L \mid x \leq \bigvee Y$ for some finite set $Y \subseteq X\}$ is the* ideal generated by X. *An ideal is* principal *if it is generated by a singleton set. Dually, the set $\lfloor X \rfloor := \{x \in L \mid \bigwedge Y \leq x$ for some finite set $Y \subseteq X\}$, is the* filter generated by X. *A filter is* principal *if it is generated by a singleton set.*[1]

Exercise 3. *Let L be a lattice, and $X \subseteq L$ be non-empty.*

1. *Verify that $\lceil X \rceil$ (resp. $\lfloor X \rfloor$) is an ideal (resp. a filter) of L;*

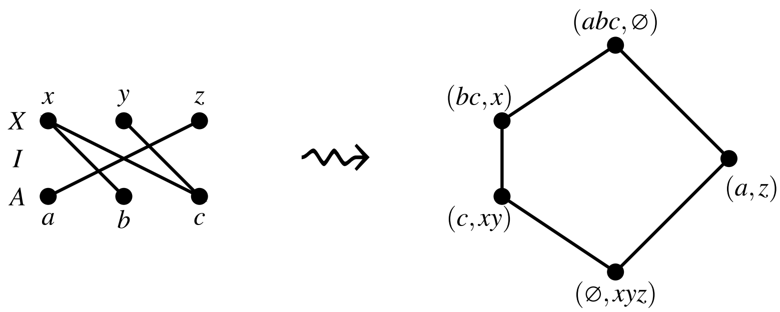
2. *Prove that $\lceil X \rceil = \bigcap\{J \mid J$ is an ideal and $X \subseteq J\}$ (resp. $\lfloor X \rfloor = \bigcap\{G \mid G$ is a filter and $X \subseteq G\}$).*

Exercise 4. *Prove that in a finite lattice, every ideal (resp. filter) is principal. Hint: for any ideal J consider $\bigvee J$ (dually for a filter).*

[1] To simplify notations for principal ideals (resp. filters), we write $\lceil a \rceil$ for $\lceil \{a\} \rceil$ (resp. $\lfloor a \rfloor$ for $\lfloor \{a\} \rfloor$).

2 Exercises from Lecture 3

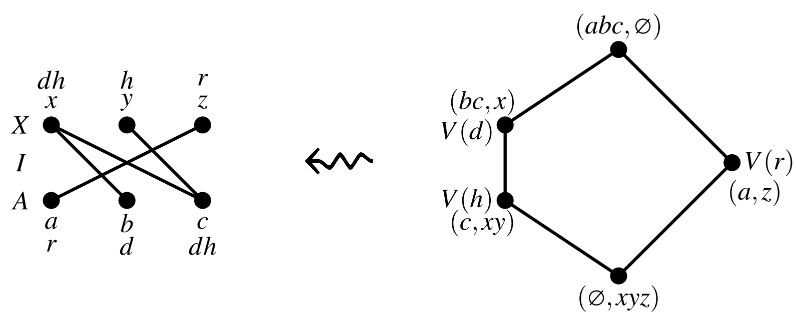
Exercise 5. *Consider the polarity* $\mathbb{P} = (A, X, I)$*, representing a 'database' of theatrical plays the set of objects of which is* $A := \{a, b, c\}$*, where* a *is* A Midsummer Night's Dream*,* b *is* King Lear*, and* c *is* Julius Caesar*, while its set of features is* $X := \{x, y, z\}$*, where* x *is* 'no happy end'*,* y *is* 'some characters are real historical figures'*, and* z *is* 'two characters fall in love with each other'*. The following picture represents* $\mathbb{P}$ *and its associated concept lattice* $\mathbb{P}^+$*.*



Let Prop $:= \{r, d, h\}$ *be the set of atomic concept-variables, where* r *stands for 'romantic comedy',* d *for 'drama' and* h *for 'historical drama'. Consider the assignment* $V :$ Prop $\to \mathbb{P}^+$ *which maps* r *to* (a, z)*,* d *to* (bc, x) *and* h *to* (c, xy)*. Then, for every* $p \in$ Prop $:= \{r, d, h\}$*, the clauses*

$$\begin{array}{lll} \mathbb{M}, a \Vdash p & \textit{iff} & a \in [\![p]\!]_{\mathbb{M}} \\ \mathbb{M}, x \succ p & \textit{iff} & x \in (\![p]\!)_{\mathbb{M}} \end{array}$$

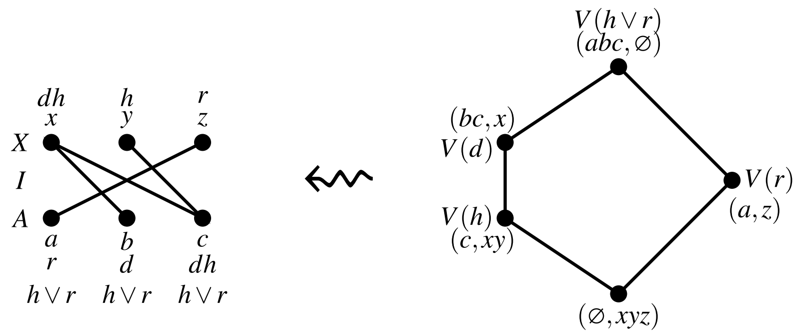
instantiate as shown in the picture below:



The valuation V *can be homomorphically extended to an interpretation map of* $\mathcal{L}$*-formulas, also denoted* V*, defined as follows:*

$$\begin{array}{rcl} V(p) & = & ([\![p]\!], (\![p]\!)) \\ V(\top) & = & (A, A^{\uparrow}) \\ V(\bot) & = & (X^{\downarrow}, X) \\ V(\varphi \wedge \psi) & = & ([\![\varphi]\!] \cap [\![\psi]\!], ([\![\varphi]\!] \cap [\![\psi]\!])^{\uparrow}) \\ V(\varphi \vee \psi) & = & (((\![\varphi]\!) \cap (\![\psi]\!))^{\downarrow}, (\![\varphi]\!) \cap (\![\psi]\!)) \end{array}$$

This translates into the possibility of extending the "membership" and "description" relations $\Vdash$ *and* $\succ$ *to* $h \vee r$ *as illustrated in the following picture:*



Verify that, if $\mathbb{M} = (\mathbb{P}, V)$ *is the model for the basic lattice logic described above, then*

$$\mathbb{M} \not\models d \wedge (h \vee r) \vdash (d \wedge h) \vee (d \wedge r) \quad \textit{and} \quad \mathbb{M} \not\models (h \vee d) \wedge (h \vee r) \vdash h \vee (r \wedge d).$$

Exercise 6. *Prove that the axioms and rules of the basic logic of formal concepts are valid in any formal context. Conclude that the basic logic of formal concepts is sound w.r.t. the class of formal contexts.*

Exercise 7. *Let* $\mathbf{L}$ *be the Lindenbaum-Tarski algebra of basic lattice logic, and let* V *be the* canonical valuation*, i.e. the one such that* $[\![p]\!]$ *(resp.* $(\![p]\!)$*) is the set of the filters (resp. ideals) of* $\mathbf{L}$ *to which* p *belongs, and let* $\mathbb{M} = (\mathbb{P}_{\mathbf{L}}, V)$ *be the canonical model (cf. Lecture 3, p5.). Verify that if* $[\![p]\!] := \{a \in A \mid p \in a\}$ *and* $(\![p]\!) := \{x \in X \mid p \in x\}$ *for every* $p \in \mathsf{Prop}$*, then* $[\![p]\!]^{\uparrow} = (\![p]\!)$ *and* $(\![p]\!)^{\downarrow} = [\![p]\!]$*. Deduce that* $V : \mathsf{Prop} \to \mathbb{P}_{\mathbf{L}}^{+}$ *is well defined.*

Exercise 8. *Prove the missing cases in the proof of the truth lemma of Lecture 3.*