Logical foundations of categorization theory
Lecture 2

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Abstract

In the first part this lecture, we introduce the basic definitions and facts on closure operators, closure systems and complete lattices; in the second part of the lecture, we expand on properties of closure operators and their link with complete lattices, in the form of Birkhoff's representation theorem for complete lattices as concept lattices arising from formal contexts.

1 Background on mathematical approach

Closure operators and complete lattices

Definition 1. *If $(P, \leq)$ is a partially ordered set, a map $c : P \to P$ is a* closure operator *on $(P, \leq)$ if, for all x, y in P:*

(a) $x \leq c(x)$ *(inflationary);*

(b) *if $x \leq y$ then $c(x) \leq c(y)$ (monotone);*

(c) $c(c(x)) = c(x)$ *(idempotent).*

If c is a closure operator on $(P, \leq)$, let $\mathscr{C}_c := \{x \in P \mid x = c(x)\}$.

Definition 2. *A bounded* lattice *is a partially ordered set $(L, \leq)$ such that:*

(a) *there exists an element $\top \in L$ such that $a \leq \top$ for every $a \in L$;*

(b) *there exists an element $\bot \in L$ such that $\bot \leq a$ for every $a \in L$;*

(c) *for any $a, b \in L$ there exists an element $a \wedge b$ such that $a \wedge b \leq a$ and $a \wedge b \leq b$ and for every $c \in L$, if $c \leq a$ and $c \leq b$ then $c \leq a \wedge b$;*

(d) *for any $a, b \in L$ there exists an element $a \vee b$ such that $a \leq a \vee b$ and $b \leq a \vee b$ and for every $c \in L$, if $a \leq c$ and $b \leq c$ then $a \vee b \leq c$.*

A lattice is complete *if*

(a') *for every $X \subseteq L$ there exists an element $\bigwedge X$ such that $\bigwedge X \leq x$ for every $x \in X$ and for every $c \in L$, if $c \leq x$ for every $x \in X$ then $c \leq \bigwedge X$ ($\bigwedge X$ is "the greatest lower bound of X");*

(b') *for every $X \subseteq L$ there exists an element $\bigvee X$ such that $x \leq \bigvee X$ for every $x \in X$ and for every $c \in L$, if $x \leq c$ for every $x \in X$ then $\bigvee X \leq c$ ($\bigvee X$ is "the least upper bound of X").*

A partial order for which (a) and (c) hold is a meet-semilattice*; a partial order for which (b) and (d) hold is a* join-semilattice*.*

A partial order for which (c') hold is a complete meet-semilattice*; a partial order for which (d') holds is a* complete join-semilattice*.*

Exercise 1. *For any partial order $(L, \leq)$,*

1. *Prove that if $X \subseteq Y \subseteq L$ then $\bigvee X \leq \bigvee Y$ and $\bigwedge Y \leq \bigwedge X$ whenever they exist;*
2. *Deduce that if $\top$ exists then $\top = \bigwedge \varnothing$ and if $\bot$ exists then $\bot = \bigvee \varnothing$.*

Exercise 2. *Prove that if $(L, \leq)$ is a complete meet-semilattice, then $(L, \leq)$ is a complete lattice. (Hint: Let $X \subseteq L$; to show that $\bigvee X$ exists, let $Y := \{a \in L \mid \forall x(x \in X \Rightarrow x \leq a)\}$...)*

Closure operators on complete lattices

Fact 1. *If c is a closure operator on a complete lattice $(L, \leq)$, and $X \subseteq \mathscr{C}_c := \{x \in P \mid x = c(x)\}$, then $\bigwedge X \in \mathscr{C}_c$.*

Proof. Clearly $\bigwedge X \leq c(\bigwedge X)$; for the converse inequality, by definition we have $\bigwedge X \leq x$ for every $x \in X$, which implies that $c(\bigwedge X) \leq c(x)$ for every $x \in X$, hence $c(\bigwedge X) \leq \bigwedge\{c(x) \mid x \in X\} = \bigwedge\{x \mid x \in X\} = \bigwedge X$, as required. ◻

Fact 2. *In a complete lattice $(L, \leq)$, if $\mathscr{C} \subseteq L$ then the map $c_{\mathscr{C}} : L \to L$ defined by the assignment $a \mapsto \bigwedge\{x \in \mathscr{C} \mid a \leq x\}$ is a closure operator.*

Proof. By definition, a is a lower bound of $\{x \in \mathscr{C} \mid a \leq x\}$, and so $a \leq \bigwedge\{x \in \mathscr{C} \mid a \leq x\} = c_{\mathscr{C}}(a)$ for every $a \in L$. If $a, b \in L$ and $a \leq b$, then by transitivity $\{x \in \mathscr{C} \mid b \leq x\} \subseteq \{x \in \mathscr{C} \mid a \leq x\}$. Hence, $c(a)$, which is a lower bound of $\{x \in \mathscr{C} \mid a \leq x\}$, is also a lower bound of $\{x \in \mathscr{C} \mid b \leq x\}$, and hence $c(a) \leq \bigwedge\{x \in \mathscr{C} \mid b \leq x\} = c(b)$, as required. Finally, let us show that $c_{\mathscr{C}}(a) = c_{\mathscr{C}}(c_{\mathscr{C}}(a))$. From $a \leq c_{\mathscr{C}}(a)$ and monotonicity that we have just proven we get $c_{\mathscr{C}}(a) \leq c_{\mathscr{C}}(c_{\mathscr{C}}(a))$. To prove the converse inequality $c_{\mathscr{C}}(c_{\mathscr{C}}(a)) \leq c_{\mathscr{C}}(a)$, by definition, it is enough to show that $c_{\mathscr{C}}(c_{\mathscr{C}}(a))$ is a lower bound of $\{x \in \mathscr{C} \mid a \leq x\}$. Let $x \in \mathscr{C}$ such that $a \leq x$; then $c_{\mathscr{C}}(a) = \bigwedge\{x \in \mathscr{C} \mid a \leq x\} \leq x$; hence by monotonicity, $c_{\mathscr{C}}(c_{\mathscr{C}}(a)) \leq c_{\mathscr{C}}(x)$. Hence to finish the proof it is enough to show that $c_{\mathscr{C}}(x) = x$. This immediately follows from the fact that $x \in \mathscr{C}$ and $\leq$ is reflexive. ◻

Definition 3. *If $(L, \leq)$ is a complete lattice, a* closure system *of $(L, \leq)$ is a subset $\mathscr{C} \subseteq L$ such that, for every $X \subseteq L$, if $X \subseteq \mathscr{C}$ then $\bigwedge X \in \mathscr{C}$.*

Closure systems on a complete lattice $(L, \leq)$ is are exactly the complete sub meet-semilattices of $(L, \leq)$, and hence they are complete lattices by the previous exercise.

Exercise 3. *Prove that, if $\mathscr{C} \subseteq L$ is a closure system of $(L, \leq)$, then for every $Y \subseteq \mathscr{C}$, $\bigvee_{\mathscr{C}} Y = c_{\mathscr{C}}(\bigvee Y)$.*

Proposition 1. *If $(L, \leq)$ is a complete lattice and*

1. *c is a closure operator on $(L, \leq)$ then $c = c_{\mathscr{C}_c}$.*
2. *$\mathscr{C} \subseteq L$ is a closure system of $(L, \leq)$, then $\mathscr{C} = \mathscr{C}_{c_{\mathscr{C}}}$.*

Proof. 1. Let $a \in L$. Since c is a closure operator, $c(c(a)) = c(a)$, hence $c(a) \in \mathscr{C}_c = \{x \in L \mid c(x) = x\}$, and $a \leq c(a)$, so $c(a) \in \{x \in \mathscr{C}_c \mid a \leq x\}$, hence $c_{\mathscr{C}_c}(a) = \bigwedge\{x \in \mathscr{C}_c \mid a \leq x\} \leq c(a)$; to prove the converse inequality $c(a) \leq c_{\mathscr{C}_c}(a) = \bigwedge\{x \mid c(x) = x$ and $a \leq x\}$, it is enough to show that $c(a)$ is a lower bound for $\{x \mid c(x) = x$ and $a \leq x\}$. Indeed, if x is such that $c(x) = x$ and $a \leq x$, then $c(a) \leq c(x) = x$, as required.

2. By definition, $\mathscr{C}_{c_{\mathscr{C}}} = \{x \in L \mid c_{\mathscr{C}}(x) = x\} = \{x \in L \mid x = \bigwedge\{y \in \mathscr{C} \mid x \leq y\}\}$. Since $\mathscr{C}$ is closed under taking the meet of any of its subsets, $\bigwedge\{y \in \mathscr{C} \mid x \leq y\} \in \mathscr{C}$, and hence $\mathscr{C}_{c_{\mathscr{C}}} \subseteq \mathscr{C}$. To prove the converse inclusion, let us show that if $x \in \mathscr{C}$, then $x = \bigwedge\{y \in \mathscr{C} \mid x \leq y\}$. Indeed, by definition, x is a lower bound of $\{y \in \mathscr{C} \mid x \leq y\}$, and hence $x \leq \bigwedge\{y \in \mathscr{C} \mid x \leq y\}$. On the other hand, since $x \in \mathscr{C}$ and $x \leq x$, we also have that $x \in \{y \in \mathscr{C} \mid x \leq y\}$, and since $\bigwedge\{y \in \mathscr{C} \mid x \leq y\}$ is a lower bound of $\{y \in \mathscr{C} \mid x \leq y\}$ by definition, we also have $\bigwedge\{y \in \mathscr{C} \mid x \leq y\} \leq x$, as required. $\square$

Hence, on any complete lattice $(L, \leq)$, there is a perfect correspondence between closure operators on $(L, \leq)$ and closure systems of $(L, \leq)$.

Closure operators and Galois connections.

Definition 4. *Let $(P, \leq)$ and $(Q, \preceq)$ be partial orders. A* Galois connection *is a pair of maps $\triangleright : P \to Q$ and $\blacktriangleright : Q \to P$ such that for every $x \in P$ and every $y \in Q$,*

$$x \leq \blacktriangleright y \quad \textit{iff} \quad y \preceq \triangleright x.$$

Exercise 4. *Prove that, in any Galois connection as above,*

1. *$x \leq \blacktriangleright\triangleright x$ and $y \leq \triangleright\blacktriangleright y$;*
2. *$x \leq x'$ implies $\triangleright x' \preceq \triangleright x$, and $y \preceq y'$ implies $\blacktriangleright y' \leq \blacktriangleright y$;*
3. *$\triangleright\blacktriangleright\triangleright x = \triangleright x$ and $\blacktriangleright\triangleright\blacktriangleright y = \blacktriangleright y$.*

Deduce from the previous items that for every Galois connection as above, $\blacktriangleright\triangleright$ is a closure operator on $(P, \leq)$ and $\triangleright\blacktriangleright$ is a closure operator on $(Q, \preceq)$.

2 Examples of closure operators

Example/Exercise 1. *A topological space is a tuple (X,τ) such that X is a set and τ is a collection of subsets of X, called* open sets*, such that $\varnothing, X \in \tau$ and τ is closed under finite intersection and arbitrary unions. Prove that the set $\mathscr{C}_\tau := \{A^c \mid A \in \tau\}$ is a closure system. Then the map $c : \mathscr{P}(X) \to \mathscr{P}(X)$ defined by the assignment $Y \mapsto \bigcap\{C \in \mathscr{C}_\tau \mid Y \subseteq C\}$ is a closure operator on $(\mathscr{P}(X), \subseteq)$. Moreover, $\mathscr{C}_\tau$ is a complete sub $\bigcap$-semilattice of the complete lattice $(\mathscr{P}(X), \subseteq)$, and for every collection of closed subsets $\mathscr{Y} \subseteq \mathscr{C}_\tau$, we have $\bigvee_{\mathscr{C}_\tau} \mathscr{Y} = c_{\mathscr{C}_\tau}(\bigcup \mathscr{Y})$.*

Example/Exercise 2. *The Polish school in logic defines a* logical system *as a tuple $\mathscr{S} = (\mathbf{Fm}, \vdash)$ such that $\mathbf{Fm}$ is the algebra of $\mathscr{L}$-formulas over a given set Φ of propositional variables for a given algebraic signature $\mathscr{L}$, and $\vdash$ is a* consequence relation on $\mathbf{Fm}$*, i.e. $\vdash \subseteq \mathscr{P}(\mathbf{Fm}) \times \mathbf{Fm}$ such that for all $\varphi, \psi \in \mathbf{Fm}$ and all $\Delta, \Gamma \subseteq \mathbf{Fm}$:*

(a) if $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$;

(b) if $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \varphi$;

(c) if $\Gamma \vdash \psi$ for every $\psi \in \Delta$ and $\Delta \vdash \varphi$, then $\Gamma \vdash \varphi$.

For any logical system $\mathscr{S}$ as above, prove that the map $c_\vdash : \mathscr{P}(\mathbf{Fm}) \to \mathscr{P}(\mathbf{Fm})$ defined by the assignment $\Gamma \mapsto \{\varphi \mid \Gamma \vdash \varphi\}$ is a closure operator on $(\mathscr{P}(\mathbf{Fm}), \subseteq)$, and its associated closure system is $\mathscr{C}_{c_\vdash} = \{\Gamma \subseteq \mathbf{Fm} \mid \forall \varphi(\Gamma \vdash \varphi \Rightarrow \varphi \in \Gamma)\}$. Which set of formulas is $\bot_{\mathscr{C}_{c_\vdash}}$?

Example/Exercise 3. *A* metric space *is an ordered pair (X,d) such that X is a set and $d : X \times X \to \mathbb{R}$ such that for all $x,y,z \in M$,*

(a) $d(x,y) = 0$ iff $x = y$ (identity of indiscernibles);

(b) $d(x,y) = d(y,x)$ (symmetry);

(c) $d(x,z) \leq d(x,y) + d(y,z)$ (subadditivity or triangle inequality).

Then $d(x,y) \geq 0$ for any $x,y \in M$ (prove it. Hint: if $2 \cdot d(x,y) \geq 0$ then $d(x,y) \geq 0$). The most important examples of metric spaces are Euclidean spaces where $X = \mathbb{R}^n$ and for every $\overline{p}, \overline{q} \in \mathbb{R}^n$,

$$d(\overline{p}, \overline{q}) := \sqrt{\Sigma_{i=1}^n (p_i - q_i)^2}$$

In a Euclidean space $\mathbb{R}^n$, a subset $Y \subseteq \mathbb{R}^n$ is convex *if for any $\overline{p}, \overline{q} \in Y$ the segment the extremes of which are $\overline{p}\ \overline{q}$ is contained in Y. Prove that the set $\mathsf{Conv}(\mathbb{R}^n)$ of the convex sets of $\mathbb{R}^n$ is a closure system of $(\mathscr{P}(\mathbb{R}^n), \subseteq)$, hence the map associating any subset $Y \subseteq \mathbb{R}^n$ with its convex hull (i.e. the smallest convex set containing Y) is a closure operator on $(\mathscr{P}(\mathbb{R}^n), \subseteq)$.*

Example/Exercise 4. *A* polarity *or* formal context *is a triple $\mathbb{P} = (A, X, I)$ such that A and X are sets and $I \subseteq A \times X$. Every polarity induces the pair of maps*

$$(\cdot)^\uparrow : \mathscr{P}(A) \to \mathscr{P}(X) \quad \textit{and} \quad (\cdot)^\downarrow : \mathscr{P}(X) \to \mathscr{P}(A),$$

respectively defined by the assignments

$$B^{\uparrow} := \{x \in X \mid \forall a(a \in B \Rightarrow aIx)\} \quad \textit{and} \quad Y^{\downarrow} := \{a \in A \mid \forall x(x \in Y \Rightarrow aIx)\}.$$

Show that for every $B \subseteq A$ and every $Y \subseteq B$,

$$B \subseteq Y^{\downarrow} \quad \textit{iff} \quad Y \subseteq B^{\uparrow}$$

i.e. form a Galois connection (cf. Definition 4), and hence induce the closure operators[1] *$(\cdot)^{\uparrow\downarrow} : \mathscr{P}(A) \to \mathscr{P}(A)$ and $(\cdot)^{\downarrow\uparrow} : \mathscr{P}(X) \to \mathscr{P}(X)$. Moreover, the fixed points of these closure operators form complete sub-$\bigcap$-semilattices of $\mathscr{P}(A)$ and $\mathscr{P}(X)$ (and hence complete lattices) respectively, which are dually isomorphic to each other via the restrictions of the maps $(\cdot)^{\uparrow}$ and $(\cdot)^{\downarrow}$ (cf. Exercise 4.3).*

This motivates the following

Definition 5. *For every formal context $\mathbb{P} = (A, X, I)$, a* formal concept *of $\mathbb{P}$ is a pair $c = (B, Y)$ such that $B \subseteq A$, $Y \subseteq X$, and $B^{\uparrow} = Y$ and $Y^{\downarrow} = B$. The set B is the* extension *of c, which we will sometimes denote $[\![c]\!]$, and Y is the* intension *of c, sometimes denoted $(\![c]\!)$. Let $L(\mathbb{P})$ denote the set of the formal concepts of $\mathbb{P}$. Then the* concept lattice *of $\mathbb{P}$ is the complete lattice*

$$\mathbb{P}^{+} := (L(\mathbb{P}), \bigwedge, \bigvee),$$

where for every $\mathscr{X} \subseteq L(\mathbb{P})$,

$$\bigwedge \mathscr{X} := (\bigcap_{c \in \mathscr{X}} [\![c]\!], (\bigcap_{c \in \mathscr{X}} [\![c]\!])^{\uparrow}) \quad \textit{and} \quad \bigvee \mathscr{X} := ((\bigcap_{c \in \mathscr{X}} (\![c]\!))^{\downarrow}, \bigcap_{c \in \mathscr{X}} (\![c]\!)).$$

Then clearly, $\top^{\mathbb{P}^{+}} := \bigwedge \varnothing = (A, A^{\uparrow})$ and $\bot^{\mathbb{P}^{+}} := \bigvee \varnothing = (X^{\downarrow}, X)$, and the partial order underlying this lattice structure is defined as follows: for any $c, d \in L(\mathbb{P})$,

$$c \leq d \quad \textit{iff} \quad [\![c]\!] \subseteq [\![d]\!] \quad \textit{iff} \quad (\![d]\!) \subseteq (\![c]\!).$$

3 Birkhoff's representation theorem of complete lattices

The following theorem, known as *Birkhoff's representation theorem of complete lattices*, is the order-theoretic foundation of Formal Concept Analysis.

Theorem 1. *Any complete lattice $\mathbb{L}$ is isomorphic to the concept lattice $\mathbb{P}^{+}$ of some formal context $\mathbb{P}$.*

Proof. If $\mathbb{L} = (L, \leq)$ is a complete lattice, then let $\mathbb{P} := (L, L, \leq)$. We want to show that $\mathbb{L}$ is isomorphic to $\mathbb{P}^{+}$. For every $a \in L$, let us show the preliminary claim that

$$a^{\uparrow\downarrow} = \{b \in L \mid b \leq a\}.$$

Indeed by definition, $a^{\uparrow} = \{x \in L \mid a \leq x\}$; then $a^{\uparrow\downarrow} = \{b \in L \mid \forall x(x \in a^{\uparrow} \Rightarrow b \leq x)\} = \{b \in L \mid \forall x(a \leq x \Rightarrow b \leq x)\}$. Hence, by transitivity, if $b \leq a$ then, for every $x \in L$,

[1] When $B = \{a\}$ (resp. $Y = \{x\}$) we write $a^{\uparrow\downarrow}$ for $\{a\}^{\uparrow\downarrow}$ (resp. $x^{\downarrow\uparrow}$ for $\{x\}^{\downarrow\uparrow}$).

if $a \leq x$ then $b \leq x$, which shows that $\{b \in L \mid b \leq a\} \subseteq a^{\uparrow\downarrow}$. To prove the converse inclusion $a^{\uparrow\downarrow} \subseteq \{b \in L \mid b \leq a\}$, we proceed by contraposition: if $b \notin \{b \in L \mid b \leq a\}$, i.e. $b \nleq a$ then there exists an element of $x \in a^{\uparrow}$, namely $x = a$ itself, such that $b \nleq x$, and hence $b \notin a^{\uparrow\downarrow}$, which finishes the proof of the preliminary claim.

Let us define the map $f : \mathbb{L} \to \mathbb{P}^+$ by the assignment $a \mapsto (a^{\uparrow\downarrow}, a^{\uparrow})$, and the map $g : \mathbb{P}^+ \to \mathbb{L}$ by the assignment $([\![c]\!], (\![c]\!)) \mapsto \bigvee [\![c]\!]$.

To show that f is surjective, let us show that for every $c = ([\![c]\!], (\![c]\!)) \in \mathbb{P}^+$,

$$c = f(g(c)).$$

For every $c = ([\![c]\!], (\![c]\!)) \in \mathbb{P}^+$,

$$f(g(c)) = f(\bigvee [\![c]\!]) = ((\bigvee [\![c]\!])^{\uparrow\downarrow}, (\bigvee [\![c]\!])^{\uparrow}),$$

hence to finish the proof that $f(g(c)) = c$, it is enough to show that $(\![c]\!) = (\bigvee [\![c]\!])^{\uparrow})$. By definition, $(\![c]\!) = [\![c]\!]^{\uparrow}$, so we need to show that $[\![c]\!]^{\uparrow} = (\bigvee [\![c]\!])^{\uparrow}$.

$$\begin{array}{rcll}
[\![c]\!]^{\uparrow} & = & \{y \in L \mid \forall a (a \in [\![c]\!] \Rightarrow a \leq y)\} & \text{definition of } [\![c]\!]^{\uparrow} \\
& = & \{y \in L \mid \bigvee [\![c]\!] \leq y\} & \bigvee \text{ least upper bound} \\
& = & (\bigvee [\![c]\!])^{\uparrow} & \text{definition of } (\bigvee [\![c]\!])^{\uparrow}
\end{array}$$

Let us show that f is an order-embedding, i.e. that for every $a, b \in L$,

$$a \leq_{\mathbb{L}} b \quad \text{iff} \quad f(a) \leq_{\mathbb{P}^+} f(b).$$

By definition, $f(a) \leq_{\mathbb{P}^+} f(b)$ iff $[\![f(a)]\!] = a^{\uparrow\downarrow} \subseteq b^{\uparrow\downarrow} = [\![f(b)]\!]$, so $a \leq_{\mathbb{L}} b$ implies that $a^{\uparrow\downarrow} \subseteq b^{\uparrow\downarrow}$ since the composition of order-reversing maps (cf. Exercise 4.2) is order preserving. For the converse implication, assume that $a^{\uparrow\downarrow} \subseteq b^{\uparrow\downarrow}$; by the preliminary claim, this is equivalent to $\{x \in L \mid x \leq a\} \subseteq \{x \in L \mid x \leq b\}$; hence $a \in \{x \in L \mid x \leq b\}$, i.e. $a \leq b$, as required. Summing up, we have showed that f is an order-isomorphism (i.e. a surjective order-embedding) between the complete lattices $\mathbb{L}$ and $\mathbb{P}^+$. Hence these two lattices are isomorphic. □

Exercise 5. *Prove that for the maps f and g defined in the proof above, $a = g(f(a))$.*