Logical foundations of categorization theory
Handout 2

Fei Liang and Alessandra Palmigiano

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1 Order-theoretic notions

Definition 1 (Maximal and minimal elements)**.** *Let $(P, \leq)$ be a partially ordered set and let $Q \subseteq P$. Then:*

(a) *$a \in Q$ is a* maximal element of *Q if, for all $x \in P$, if $a \leq x$ and $x \in Q$ then $a = x$;*

(b) *$a \in Q$ is a* minimal element of *Q if, for all $x \in P$, if $x \leq a$ and $x \in Q$ then $a = x$;*

(c) *$a \in Q$ is the* greatest (or maximum) element of *Q if $x \leq a$ for all $x \in Q$;*

(d) *$a \in Q$ is the* least (or minimum) element of *Q if $a \leq x$ for all $x \in Q$.*

We denote the set of maximal elements of Q by $MaxQ$ and the set of minimal elements of Q by $MinQ$.

Exercise 1. *Prove that if $y \in Q$ is the maximum element of Q, then $MaxQ = \{y\}$; dually, if $y \in Q$ the minimum element of Q, then $MinQ = \{y\}$. Deduce that if the maximum (or minimum) element of Q exists, then it is unique.*

Definition 2 (Upper bound and lower bound)**.** *Let $(P, \leq)$ be a partially ordered set and let $Q \subseteq P$. Then:*

(a) *$a \in P$ is an* upper bound of *Q if $x \leq a$ for all $x \in Q$;*

(b) *$a \in P$ is a* lower bound of *Q if $a \leq x$ for all $x \in Q$;*

The set of all upper bounds of Q is denoted by Q^u and the set of all lower bounds by Q^ℓ. If the minimum element (denoted by $\bigvee Q$) of Q^u exists, then $\bigvee Q$ is called the least upper bound (or supremum) *of Q. Dually, if the maximum element (denoted by $\bigwedge Q$) of Q^ℓ exists, then $\bigwedge Q$ is called the* greatest lower bound (or infimum) *of Q.*

Exercise 2. *Give an example of a partially ordered set $(P, \leq)$ and $Q \subseteq P$ in which $\bigvee Q \notin Q$ and an example of a partially ordered set $(P, \leq)$ and $Q \subseteq P$ in which $\bigwedge Q \notin Q$.*

2 Lattices and closure systems

Definition 3 (Bounded lattice)**.** *A* lattice *is a partially ordered set* $(L,\leq)$ *such that:*

(a) the infimum $a\wedge b$ *of the set* $\{a,b\}$ *exists for any* $a,b\in L$*;*

(b) the supremum $a\vee b$ *of the set* $\{a,b\}$ *exists for any* $a,b\in L$*;*

(c) the maximum $\top$ *of* L *exists;*

(d) the minimum $\bot$ *of* L *exists.*

Such a lattice is complete *if*

(a') $\bigwedge X\in L$ *exists for every* $X\subseteq L$*;*

(b') $\bigvee X\in L$ *exists for every* $X\subseteq L$*.*

A partial order for which (a) and (c) hold is a meet-semilattice*; a partial order for which (b) and (d) hold is a* join-semilattice*. A partial order for which (a') hold is a* complete meet-semilattice*; a partial order for which (b') holds is a* complete join-semilattice*.*

Exercise 3. *For any partial order* $(P,\leq)$*,*

1. *Prove that if* $X\subseteq Y\subseteq P$ *then* $\bigvee X\leq\bigvee Y$ *and* $\bigwedge Y\leq\bigwedge X$ *whenever they exist;*
2. *Deduce that if* $\top$ *exists then* $\top=\bigwedge\varnothing$ *and if* $\bot$ *exists then* $\bot=\bigvee\varnothing$*.*

Exercise 4. *Prove that if* $(L,\leq)$ *is a complete meet-semilattice, then* $(L,\leq)$ *is a complete lattice. Hint: Let* $X\subseteq L$*; to show that* $\bigvee X$ *exists, consider* X^u *(cf. Definition* **??***).*

Definition 4 (Closure system)**.** *If* $(L,\leq)$ *is a complete lattice, a* closure system *of* $(L,\leq)$ *is a subset* $\mathscr{C}\subseteq L$ *such that, for every* $X\subseteq L$*, if* $X\subseteq\mathscr{C}$ *then* $\bigwedge X\in\mathscr{C}$*.*

Exercise 5. *Let* $(L,\leq)$ *be a complete lattice. Prove that* $\top\in\mathscr{C}$ *for any closure system of* $(L,\leq)$*.*

Exercise 6. *For any topological space* (X,τ)*, show that the set* $\mathscr{C}_\tau:=\{A^c\mid A\in\tau\}$ *is a closure system, where* A^c *is the relative complement of* A *with respect to* X*.*

Exercise 7. *For any logical system* $\mathscr{S}$*, let the map* $c_\vdash:\mathscr{P}(\mathbf{Fm})\to\mathscr{P}(\mathbf{Fm})$ *be defined by the assignment* $\Gamma\mapsto\{\varphi\mid\Gamma\vdash\varphi\}$*. Show that* $\mathscr{C}_{c_\vdash}=\{\Gamma\subseteq\mathbf{Fm}\mid\forall\varphi(\Gamma\vdash\varphi\Rightarrow\varphi\in\Gamma)\}$ *is a closure system. Which set of formulas is* $\bot_{\mathscr{C}_{c_\vdash}}$*?*

Exercise 8. *Let* $(L,\leq)$ *be a complete lattice. Prove that, if* $\mathscr{C}\subseteq L$ *is a closure system of* $(L,\leq)$*, then for every* $Y\subseteq\mathscr{C}$*,* $\bigvee_{\mathscr{C}}Y=c_{\mathscr{C}}(\bigvee Y)$*, where the map* $c_{\mathscr{C}}:L\to L$ *is defined by the assignment* $a\mapsto\bigwedge\{x\in\mathscr{C}\mid a\leq x\}$*. (Notice that:* $\bigvee_{\mathscr{C}}Y$ *is the supremum of* Y *in* $\mathscr{C}$ *(cf. Exercise 5) and* $\bigvee Y$ *is the supremum of* Y *in* L*.)*

3 Birkhoff's representation theorem for complete lattices

Definition 5 (Homomorphism)**.** *Let L and K be bounded lattices. A map $f : L \to K$ is said to be a* homomorphism *(or bounded lattice homomorphism) if*

(a) for all $a, b \in L$, $f(a \wedge_L b) = f(a) \wedge_K f(b)$;

(b) for all $a, b \in L, f(a \vee_L b) = f(a) \vee_K f(b)$;

(c) $f(\top_L) = \top_K$;

(d) $f(\bot_L) = \bot_K$.

If L and K are complete lattices, $f : L \to K$ is said to be a complete homomorphism *if*

(a') for any set $X \subseteq L$, $f(\bigwedge_L X) = \bigwedge_K \{f(a) \mid a \in L\}$;

(b') for any set $X \subseteq L$, $f(\bigvee_L X) = \bigvee_K \{f(a) \mid a \in L\}$.

f is said to be an isomorphism *(or lattice isomorphism) if f is bijection.*

Exercise 9. *Let L and K be bounded lattices. Show that $f : L \to K$ is an isomorphism iff it is surjective and is an order-embedding.* [1]

Definition 6. *For every formal context $\mathbb{P} = (A, X, I)$, a* formal concept *of $\mathbb{P}$ is a pair $c = (B, Y)$ such that $B \subseteq A$, $Y \subseteq X$, and $B^{\uparrow} = Y$ and $Y^{\downarrow} = B$ (cf. see Definition Exercise/Example 4 in Lecture 2). The set B is the* extension *of c, which we will sometimes denote $[\![c]\!]$, and Y is the* intension *of c, sometimes denoted $(\![c]\!)$. Let $L(\mathbb{P})$ denote the set of the formal concepts of $\mathbb{P}$. Then the* concept lattice *of $\mathbb{P}$ is the complete lattice*

$$\mathbb{P}^+ := (L(\mathbb{P}), \bigwedge, \bigvee),$$

where for every $\mathscr{X} \subseteq L(\mathbb{P})$,

$$\bigwedge \mathscr{X} := (\bigcap_{c \in \mathscr{X}} [\![c]\!], (\bigcap_{c \in \mathscr{X}} [\![c]\!])^{\uparrow}) \quad \text{and} \quad \bigvee \mathscr{X} := ((\bigcap_{c \in \mathscr{X}} (\![c]\!))^{\downarrow}, \bigcap_{c \in \mathscr{X}} (\![c]\!)).$$

and the partial order underlying this lattice structure is defined as follows: for any $c, d \in L(\mathbb{P})$,

$$c \leq d \quad \text{iff} \quad [\![c]\!] \subseteq [\![d]\!] \quad \text{iff} \quad (\![d]\!) \subseteq (\![c]\!).$$

Exercise 10. *Prove that $\top^{\mathbb{P}^+} := \bigwedge \varnothing = (A, A^{\uparrow})$ and $\bot^{\mathbb{P}^+} := \bigvee \varnothing = (X^{\downarrow}, X)$.*

Exercise 11. *Compute the concept lattices associated with the polarities $\mathbb{P} = (A, X, I)$, such that $A = \{a, b, c\}$, $X = \{x, y, z\}$ and $I \subseteq A \times X$ in the following cases:*

(1) $I = \{(a, x), (b, y), (c, z)\}$;

(2) $I = \{(a, x), (b, x), (b, y), (c, y), (c, z)\}$;

(3) $I = \{(a, x), (a, y), (b, x), (b, z), (c, y), (c, z)\}$.

Exercise 12 (Birkhoff's representation theorem)**.** *Complete the proof of Birkhoff's theorem in Lecture 2.*

[1] Let $(P, \leq)$ and $(Q, \leq)$ be partially ordered sets. A map $f : P \to Q$ is an order-embedding if, for all $x, y \in P$, $x \leq y$ in P if and only if $f(x) \leq f(y)$ in Q.