Logical foundations of categorization theory
Handout 1

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1 Preliminaries

Definition 1 (Partial order)**.** *Let P be a set. A* partial order *on P is a binary relation $\leq$ on P such that, for all $x,y,z \in P$,*

(a) $x \leq x$ *(reflexive);*

(b) if $x \leq y$ *and* $y \leq x$ *then* $x = y$ *(antisymmetric);*

(c) if $x \leq y$ *and* $y \leq z$ *then* $x \leq z$ *(transitive).*

If $\leq$ is a partial order on P, we call $(P,\leq)$ a partially ordered set (poset).

Definition 2 (Closure operator)**.** *If $(P,\leq)$ is a partially ordered set, a map $c : P \to P$ is a* closure operator *on $(P,\leq)$, if for all $x,y \in P$,*

(a) $x \leq c(x)$ *(inflationary);*

(b) if $x \leq y$ *then* $c(x) \leq c(y)$ *(monotone);*

(c) $c(c(x)) = c(x)$ *(idempotent).*

If c is a closure operator on $(P,\leq)$, let $\mathscr{C}_c := \{x \in P \mid x = c(x)\}$.

2 Examples and Exercises

Topological Spaces

Definition 3. *A topological space is a tuple (X,τ) such that X is a set and τ is a collection of subsets of X, called* open sets*, such that:*

(a) $\varnothing, X \in \tau$*;*

(b) τ *is closed under finite intersection and arbitrary unions.*

Exercise 1. *Prove that, for every topological space (X,τ), the map $c : \mathscr{P}(X) \to \mathscr{P}(X)$ defined by the assignment $Y \mapsto \bigcap\{C \in \mathscr{C}_\tau \mid Y \subseteq C\}$ is a closure operator on $(\mathscr{P}(X),\subseteq)$, where $\mathscr{C}_\tau := \{A^c \mid A \in \tau\}$ and A^c is the relative complement of A with respect to X.*

Logical Systems

Definition 4. *The Polish school in logic defines a* logical system *as a tuple* $\mathscr{S} = (\mathbf{Fm}, \vdash)$ *such that* $\mathbf{Fm}$ *is the algebra of* $\mathscr{L}$*-formulas over a given set* Φ *of propositional variables for a given algebraic signature* $\mathscr{L}$*, and* $\vdash$ *is a* consequence relation on $\mathbf{Fm}$*, i.e.* $\vdash \subseteq \mathscr{P}(\mathbf{Fm}) \times \mathbf{Fm}$ *such that for all* $\varphi, \psi \in \mathbf{Fm}$ *and all* $\Delta, \Gamma \subseteq \mathbf{Fm}$*:*

(a) if $\varphi \in \Gamma$*, then* $\Gamma \vdash \varphi$*;*

(b) if $\Gamma \vdash \varphi$ *and* $\Gamma \subseteq \Delta$*, then* $\Delta \vdash \varphi$*;*

(c) if $\Gamma \vdash \psi$ *for every* $\psi \in \Delta$ *and* $\Delta \vdash \varphi$*, then* $\Gamma \vdash \varphi$*.*

Exercise 2. *For any logical system* $\mathscr{S}$ *as above, prove that the map* $c_{\vdash} : \mathscr{P}(\mathbf{Fm}) \to \mathscr{P}(\mathbf{Fm})$ *defined by the assignment* $\Gamma \mapsto \{\varphi \mid \Gamma \vdash \varphi\}$ *is a closure operator on* $(\mathscr{P}(\mathbf{Fm}), \subseteq)$*.*

Galois Connections

Definition 5 (Galois connection)**.** *Let* $(P, \leq)$ *and* $(Q, \preceq)$ *be partial orders. A* Galois connection *is a pair of maps* $\rhd : P \to Q$ *and* $\blacktriangleright : Q \to P$ *such that, for every* $x \in P$ *and every* $y \in Q$*,*

$$x \leq \blacktriangleright y \quad \text{iff} \quad y \preceq \rhd x.$$

Exercise 3. *Prove that, in any Galois connection (cf. Definition* **??***),*

1. $x \leq \blacktriangleright \rhd x$ *and* $y \leq \rhd \blacktriangleright y$*;*
2. $x \leq x'$ *implies* $\rhd x' \preceq \rhd x$*, and* $y \preceq y'$ *implies* $\blacktriangleright y' \leq \blacktriangleright y$*;*
3. $\rhd \blacktriangleright \rhd x = \rhd x$ *and* $\blacktriangleright \rhd \blacktriangleright y = \blacktriangleright y$*.*

Deduce from the previous items that for every Galois connection as above, $\blacktriangleright \rhd$ *is a closure operator on* $(P, \leq)$ *and* $\rhd \blacktriangleright$ *is a closure operator on* $(Q, \preceq)$*.*

Polarities

Definition 6. *A* polarity *or* formal context *is a triple* $\mathbb{P} = (A, X, I)$ *such that* A *and* X *are sets and* $I \subseteq A \times X$*. Every polarity induces the pair of maps*

$$(\cdot)^{\uparrow} : \mathscr{P}(A) \to \mathscr{P}(X) \quad \text{and} \quad (\cdot)^{\downarrow} : \mathscr{P}(X) \to \mathscr{P}(A),$$

respectively defined by the assignments

$$B^{\uparrow} := \{x \in X \mid \forall a(a \in B \Rightarrow aIx)\} \quad \text{and} \quad Y^{\downarrow} := \{a \in A \mid \forall x(x \in Y \Rightarrow aIx)\}.$$

Exercise 4. *Show that*

1. *the map* $\uparrow : \mathscr{P}(A) \to \mathscr{P}(X)$ *and the map* $\downarrow : \mathscr{P}(X) \to \mathscr{P}(A)$ *form a Galois connection, that is, for every* $B \subseteq A$ *and every* $Y \subseteq X$*,*

$$B \subseteq Y^{\downarrow} \quad \text{iff} \quad Y \subseteq B^{\uparrow}$$

2. *deduce that* $(\cdot)^{\uparrow\downarrow} : \mathscr{P}(A) \to \mathscr{P}(A)$ *and* $(\cdot)^{\downarrow\uparrow} : \mathscr{P}(X) \to \mathscr{P}(X)$ *are closure operators on* $(\mathscr{P}(A), \subseteq)$ *and* $(\mathscr{P}(X), \subseteq)$ *respectively.*[1]

[1] When $B = \{a\}$ (resp. $Y = \{x\}$) we write $a^{\uparrow\downarrow}$ for $\{a\}^{\uparrow\downarrow}$ (resp. $x^{\downarrow\uparrow}$ for $\{x\}^{\downarrow\uparrow}$).